Government of Tamilnadu

MATHEMATICS

X STANDARD

NOT FOR SALE

Untouchability is Inhuman and a Crime

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Preface

It is gratifying to note that education as a whole and school education in particular witness marked changes in the state of Tamil Nadu resulting in the implementation of uniform curriculum for all streams in the school education system. This is a golden opportunity given by the Government of Tamil Nadu which must be utilized for the overall improvement of education in Tamil Nadu.

Mathematics, the queen of all sciences, remains and will remain as a subject with great charm having an intrinsic value and beauty of its own. It plays an indispensable role in sciences, engineering and other subjects as well. So, mathematical knowledge is essential for the growth of science and technology, and for any individual to shine well in the field of one's choice. In addition, a rigorous mathematical training gives one not only the knowledge of mathematics but also a disciplined thought process, an ability to analyze complicated problems.

Thiruvalluvar, the prophetic Tamil poet, had as far back as at least two thousand years ago, underlined the importance and the value of mathematical education by saying,

எண்ணென்ப ஏனை எழுத்தென்ப இவ்விரண்டும்
கண்ணென்ப வாழும் உயிர்க்கு. – குறள் (392)

The two that are known as numbers and letters
They say are the eyes of people on the earth.

– Kural (392)

We need the power and prowess of mathematics to face and solve the ever increasing complex problems that we encounter in our life. Furthermore, mathematics is a supremely creative force and not just a problem solving tool. The learners will realize this fact to their immense satisfaction and advantage as they learn more and more of mathematics.

Besides, a good mathematical training is very much essential to create a good work force for posterity. The rudiments of mathematics attained at the school level form the basis of higher studies in the field of mathematics and other sciences. Besides learning the basics of mathematics, it is also important to learn how to apply them in solving problems.

Deeper understanding of basic principles and problem solving are the two important components of learning mathematics. This book is a step in this direction. It is intended to help the students to grasp the fundamentals of mathematics and apply them in problem solving. It also fosters an informed awareness of how mathematics develops and works in different situations. With this end in view, the chapters are arranged in their natural and logical order with a good number of worked out examples. Each section of a chapter is designed in such a way as to provide the students the much needed practice which promotes a thorough understanding of the concepts. We suggest that before going into the problems, the teachers and the students get themselves acquainted with the underlying mathematical ideas and their connections which precede the set of problems given in the exercises.

However, be it remembered that mathematics is more than the science of numbers. The teacher in the classroom is the most important person whose help and guidance are indispensable in learning mathematics. During the stage of transition from basic mathematics to higher mathematics, the teachers have a more significant role to play. In this context we hope that this book serves the purpose and acts as a catalyst. To reap the maximum benefit out of this, the teacher should necessarily strive for a two-way communication. This endeavour will undoubtedly pave the way for learner-centered activities in the class rooms. Moreover, this text book is aimed at giving the students a space to explore mathematics and develop skills in all directions. As we have mentioned already, there are two parts in learning mathematics. One is learning the basics and the other is applying the basics in problem solving. Going through the examples in the text does help in understanding the methods; but learning basics, solving exercise problems on one's own and then trying to create new related problems alone will help consolidate one's mathematical knowledge.

We learn Mathematics by doing Mathematics.

We would be grateful for suggestions and comments from experts, teachers and students for the improvement of this book.

-Textbook team

SYLLABUS

Topic	Content	Expected Learning Outcomes	Transactional Teaching Strategy	No. of Periods
I. Sets and Functions	i. Introduction ii. Properties of operations on sets iii. De Morgan's laws-verification using examples, Venn diagrams. iv. Formula for $n(A \cup B \cup C)$ v. Functions	• To revise the basic concepts on Set operations • To understand the properties of operations of sets - commutative, associative, and distributive restricted to three sets. • To understand the laws of complementation of sets. • To understand De Morgan's laws and demonstrating them by Venn diagram as well. • To solve word problems using the formula as well as Venn diagram. • To understand the definition , types and representation of functions. • To understand the types of functions with simple examples.	Use Venn diagrams for all illustrations Give examples of functions from economics, medicine, science etc.	26
II. Sequences and Series of Real Numbers	i. Introduction ii. Sequences iii. Arithmetic Progression (A.P) iv. Geometric Progression (G.P) v. Series	• To understand to identify an Arithmetic Progression and a Geometric Progression. • Able to apply to find the nth term of an Arithmetic Progression and a Geometric Progression. • To determine the sum of n terms of an Arithmetic Progression and a Geometric Progression. • To determine the sum of some finite series.	Use pattern approach Use dot pattern as teaching aid Use patterns to derive formulae Examples to be given from real life situations	27
III. Algebra	i. Solving linear equations ii. Polynomials iii. Synthetic division iv. Greatest Common Divisor (GCD) and Least Common Multiple (LCM) v. Rational expressions vi. Square root vii. Quadratic Equations	• To understand the idea about pair of linear equations in two unknowns. Solving a pair of linear equations in two variables by elimination method and cross multiplication method. • To understand the relationship between zeros and coefficients of a polynomial with particular reference to quadratic polynomials.	Illustrative examples – Use charts as teaching aids Recall GCD and LCM of numbers initially	

III. Algebra		• To determine the remainder and the quotient of the given polynomial using Synthetic Division Method. • To determine the factors of the given polynomial using Synthetic Division Method. • Able to understand the difference between GCD and LCM, of rational expression. • Able to simplify rational expressions (Simple Problems), • To understand square roots. • To understand the standard form of a quadratic equation . • To solve quadratic equations (only real root) - by factorization, by completing the square and by using quadratic formula. • Able to solve word problems based on quadratic equations. • Able to correlate relationship between discriminant and nature of roots. • Able to Form quadratic equation when the roots are given.	Compare with operations on fractions Compare with the square root operation on numerals. Help students visualize the nature of roots algebraically and graphically.	40
IV. Matrices	i. Introduction ii. Types of matrices iii. Addition and subtraction iv. Multiplication v. Matrix equation	• Able to identify the order and formation of matrices • Able to recognize the types of matrices • Able to add and subtract the given matrices. • To multiply a matrix by a scalar, and the transpose of a matrix. • To multiply the given matrices (2x2; 2x3; 3x2 Matrices). • Using matrix method solve the equations of two variables.	Using of rectangular array of numbers. Using real life situations. Arithmetic operations to be used	16

V. Coordinate Geometry	i. Introduction ii. Revision :Distance between two points iii. Section formula, Mid point formula, Centroid formula iv. Area of a triangle and quadrilateral v. Straight line	• To recall the distance between two points, and locate the mid point of two given points. • To determine the point of division using section formula. • To calculate the area of a triangle. • To determine the slope of a line when two points are given, equation is given. • To find an equation of line with the given information. • Able to find equation of a line in: slope-intercept form, point -slope form, two -point form, intercept form. • To find the equation of a straight line passing through a point which is (i) parallel (ii) perpendicular to a given straight line.	Simple geometrical result related to triangle and quadrilaterals to be verified as applications. the form $y = mx + c$ to be taken as the starting point	25
VI. Geometry	i. Basic proportionality theorem (with proof) ii. Converse of Basic proportionality theorem (with proof) iii. Angle bisector theorem (with proof - internal case only) iv. Converse of Angle bisector theorem (with proof - internal case only) v. Similar triangles (theorems without proof) vi. Pythagoras theorem and Tangent-Chord theorem (without proof)	• To understand the theorems and apply them to solve simple problems.	Paper folding symmetry and transformation techniques to be adopted. Formal proof to be given Drawing of figures Step by step logical proof with diagrams to be explained and discussed	20
VII. Trigonometry	i. Introduction ii. Identities iii. Heights and distances	• Able to identify the Trigonometric identities and apply them in simple problems. • To understand trigonometric ratios and applies them to calculate heights and distances. (not more than two right triangles)	By using Algebraic formulae Using trigonometric identities. The approximate nature of values to be explained	21

VIII. Mensuration	i. Introduction ii. Surface Area and Volume of Cylinder, Cone, Sphere, Hemisphere, Frustum iii. Surface area and volume of combined figures iv. Invariant volume	• To determine volume and surface area of cylinder, cone, sphere, hemisphere, frustum • Volume and surface area of combined figures (only two). • Some problems restricted to constant Volume.	Use 3D models to create combined shapes Use models and pictures ad teaching aids. Choose examples from real life situations.	24
IX. Practical Geometry	i. Introduction ii. Construction of tangents to circles iii. Construction of Triangles iv. Construction of cyclic quadrilateral	• Able to construct tangents to circles. • Able to construct triangles, given its base, vertical angle at the opposite vertex and (a) median (b) altitude • Able to construct a cyclic quadrilateral	To introduce algebraic verification of length of tangent segments. Recall related properties of angles in a circle before construction. Recall relevant theorems in theoretical geometry	15
X. Graphs	i. Introduction ii. Quadratic graphs iii. Some special graphs	• Able to solve quadratic equations through graphs • Able to apply graphs to solve word problems	Interpreting skills also to be taken care of graphs of quadratics to precede algebraic treatment. Real life situations to be introduced.	10
XI. Statistics	i. Recall Measures of central tendency ii. Measures of dispersion iii. Coefficient of variation	• To recall Mean for grouped and ungrouped data situation to be avoided). • To understand the concept of Dispersion and able to find Range, Standard Deviation and Variance. • Able to calculate the coefficient of variation.	Use real life situations like performance in examination, sports, etc.	16
XII. Probability	i. Introduction ii. Probability-theoretical approach iii. Addition Theorem on Probability	• To understand Random experiments, Sample space and Events – Mutually Exclusive, Complementary, certain and impossible events. • To understand addition Theorem on probability and apply it in solving some simple problems.	Diagrams and investigations on coin tossing, die throwing and picking up the cards from a deck of cards are to be used.	15

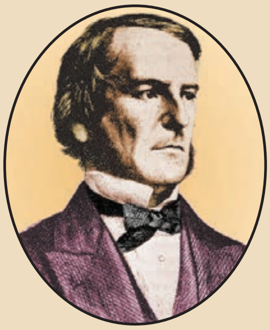
CONTENTS

1

SETS AND FUNCTIONS

A set is Many that allows itself to be thought of as a One
- Georg Cantor

- Introduction
- Sets
- Properties of set operations
- De Morgan's Laws
- Functions



GEORGE BOOLE

(1815-1864)
England

Boole believed that there was a close analogy between symbols that represent logical interactions and algebraic symbols.

He used mathematical symbols to express logical relations. Although computers did not exist in his day, Boole would be pleased to know that his Boolean algebra is the basis of all computer arithmetic.

As the inventor of Boolean logic-the basis of modern digital computer logic - Boole is regarded in hindsight as a founder of the field of computer science.

1.1 Introduction

The concept of set is one of the fundamental concepts in mathematics. The notation and terminology of set theory is useful in every part of mathematics. So, we may say that set theory is the language of mathematics. This subject, which originated from the works of George Boole (1815-1864) and Georg Cantor (1845-1918) in the later part of 19th century, has had a profound influence on the development of all branches of mathematics in the 20th century. It has helped in unifying many disconnected ideas and thus facilitated the advancement of mathematics.

In class IX, we have learnt the concept of set, some operations like union, intersection and difference of two sets. Here, we shall learn some more concepts relating to sets and another important concept in mathematics namely, function. First let us recall basic definitions with some examples. We denote all positive integers (natural numbers) by $\mathbb{N}$ and all real numbers by $\mathbb{R}$.

1.2 Sets

Definition

A set is a collection of well-defined objects. The objects in a set are called elements or members of that set.

Here, "well-defined" means that the criteria for deciding if an object belongs to the set or not, should be defined without confusion.

For example, the collection of all "tall people" in Chennai does not form a set, because here, the deciding criteria "tall people" is not clearly defined. Hence this collection does not define a set.

Notation

We generally use capital letters like A, B, X, etc. to denote a set. We shall use small letters like x, y, etc. to denote elements of a set. We write $x \in Y$ to mean x is an element of the set Y. We write $t \notin Y$ to mean t is not an element of the set Y.

Examples

(i) The set of all high school students in Tamil Nadu.

(ii) The set of all students either in high school or in college in Tamil Nadu.

(iii) The set of all positive even integers.

(iv) The set of all integers whose square is negative.

(v) The set of all people who landed on the moon.

Let A, B, C, D and E denote the sets defined in (i), (ii), (iii), (iv), and (v) respectively. Note that square of any integer is an integer that is either zero or positive and so there is no integer whose square is negative. Thus, the set D does not contain any element. Any such set is called an empty set. We denote the empty set by ϕ.

Definition

(i) A set is said to be a finite set if it contains only a finite number of elements in it.

(ii) A set which is not finite is called an infinite set.

Observe that the set A given above is a finite set, whereas the set C is an infinite set. Note that empty set contains no elements in it. That is, the number of elements in an empty set is zero. Thus, empty set is also a finite set.

Definition

(i) If a set X is finite, then we define the cardinality of X to be the number of elements in X. Cardinality of a set X is denoted by $n(X)$.

(ii) If a set X is infinite, then we denote the cardinality of X by a symbol ∞.

Now looking at the sets A, B in the above examples, we see that every element of A is also an element of B. In such cases we say A is a subset of B.

Let us recall some of the definitions that we have learnt in class IX.

Subset Let X and Y be two sets. We say X is a subset of Y if every element of X is also an element of Y. That is, X is a subset of Y if $z \in X$ implies $z \in Y$. It is clear that every set is a subset of itself.

If X is a subset of Y, then we denote this by $X \subseteq Y$.

Set Equality Two sets X and Y are said to be equal if both contain exactly same elements. In such a case, we write $X = Y$. That is, $X = Y$ if and only if $X \subseteq Y$ and $Y \subseteq X$.

Equivalent Sets Two finite sets X and Y are said to be equivalent if $n(X) = n(Y)$.

For example, let $P = \{x \mid x^2 - x - 6 = 0\}$ and $Q = \{3, -2\}$. It is easy to see that both P, Q contain same elements and so $P = Q$. If $F = \{3, 2\}$, then F, Q are equivalent sets but $Q \neq F$. Using the concept of function, one can define the equivalent of two infinite sets

Power Set Given a set A, let $P(A)$ denote the collection of all subsets of A. The set $P(A)$ is called the power set of A.

If $n(A) = m$, then the number of elements in $P(A)$ is given by $n(P(A)) = 2^m$.

For example, if $A = \{a,b,c\}$, then $P(A) = \{\phi, \{a\}, \{b\}, \{c\}. \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$ and hence $n(P(A)) = 8$.

Now, given two sets, how can we create new sets using the given sets?

One possibility is to put all the elements together from both sets and create a new set. Another possibility is to create a set containing only common elements from both sets. Also, we may create a set having elements from one set that are not in the other set. Following definitions give a precise way of formalizing these ideas. We include Venn diagram next to each definition to illustrate it.

1.3 Operations on sets

Let X and Y be two sets. We define the following new sets:

(i) **Union** $X \cup Y = \{z \mid z \in X \text{ or } z \in Y\}$

(read as "X union Y")

Note that $X \cup Y$ contains all the elements of X and all the elements of Y and the Fig. 1.1 illustrates this.

It is clear that $X \subseteq X \cup Y$ and also $Y \subseteq X \cup Y$.

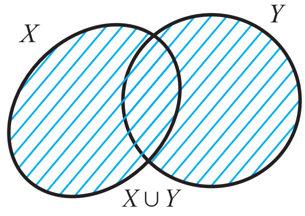


Fig. 1.1

(ii) **Intersection** $X \cap Y = \{z \mid z \in X \text{ and } z \in Y\}$

(read as "X intersection Y")

Note that $X \cap Y$ contains only those elements which belong to both X and Y and the Fig. 1.2 illustrates this.

It is trivial that $X \cap Y \subseteq X$ and also $X \cap Y \subseteq Y$.

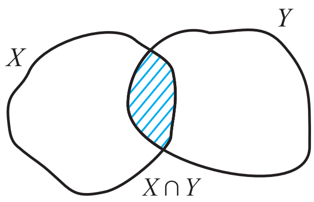


Fig. 1.2

(iii) **Set difference** $X \setminus Y = \{z \mid z \in X \text{ but } z \notin Y\}$

(read as "X difference Y")

Note that $X \setminus Y$ contains only elements of X that are not in Y and the Fig. 1.3 illustrates this. Also, some authors use $A - B$ for $A \setminus B$. We shall use the notation $A \setminus B$ which is widely used in mathematics for set difference.

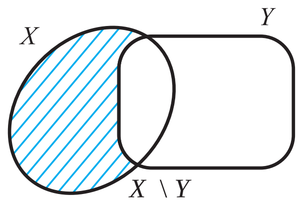


Fig. 1.3

(iv) **Symmetric Difference** $X \triangle Y = (X \setminus Y) \cup (Y \setminus X)$

(read as "X symmetric difference Y"). Note that $X \triangle Y$ contains all elements in $X \cup Y$ that are not in $X \cap Y$.

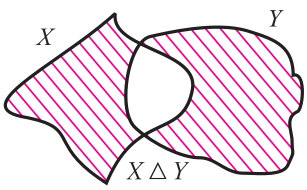


Fig. 1.4

(v) **Complement** If $X \subseteq U$, where U is a universal set, then $U \setminus X$ is called the complement of X with respect to U. If underlying universal set is fixed, then we denote $U \setminus X$ by X' and is called complement of X. The difference set $A \setminus B$ can also be viewed as the complement of B with respect to A.

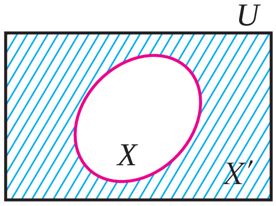


Fig. 1.5

(vi) **Disjoint sets** Two sets X and Y are said to be disjoint if they do not have any common element. That is, X and Y are disjoint if $X \cap Y = \phi$.
It is clear that $n(A \cup B) = n(A) + n(B)$ if A and B are disjoint finite sets.

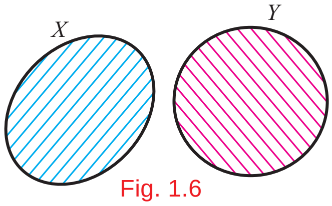


Fig. 1.6

Remarks

Usually circles are used to denote sets in Venn diagrams. However any closed curve may also be used to represent a set in a Venn diagram. While writing the elements of a set, we do not allow repetitions of elements in that set.

Now, we shall see some examples.

Let $A = \{x \mid x \text{ is a positive integer less than } 12\}$, $B = \{1, 2, 4, 6, 7, 8, 12, 15\}$ and $C = \{-2, -1, 0, 1, 3, 5, 7\}$. Now let us find the following:

(i) $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$
$= \{x \mid x \text{ is a positive integer less than } 12, \text{ or } x = 12, \text{ or } 15\}$
$= \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 15\}$.

(ii) $C \cap B = \{y \mid y \in C \text{ and } y \in B\} = \{1, 7\}$.

(iii) $A \setminus C = \{x \mid x \in A \text{ but } x \notin C\} = \{2, 4, 6, 8, 9, 10, 11\}$.

(iv) $A \triangle C = (A \setminus C) \cup (C \setminus A)$
$= \{2, 4, 6, 8, 9, 10, 11\} \cup \{-2, -1, 0\} = \{-2, -1, 0, 2, 4, 6, 8, 9, 10, 11\}$.

(v) Let $U = \{x \mid x \text{ is an integer}\}$ be the universal set.
Note that 0 is neither positive nor negative. Therefore, $0 \notin A$.

Now, $A' = U \setminus A = \{x : x \text{ is an integer but it should not be in } A\}$
$= \{x \mid x \text{ is either zero or a negative integer or positive integer greater than or equal to } 12\}$
$= \{\cdots, -4, -3, -2, -1, 0\} \cup \{12, 13, 14, 15, \cdots\}$
$= \{\cdots, -4, -3, -2, -1, 0, 12, 13, 14, 15, \cdots\}$.

Let us list out some useful results.

Let U be a universal set and A, B are subsets of U. Then the following hold:

(i) $A \setminus B = A \cap B'$ (ii) $B \setminus A = B \cap A'$

(iii) $A \setminus B = A \Leftrightarrow A \cap B = \phi$ (iv) $(A \setminus B) \cup B = A \cup B$

(v) $(A \setminus B) \cap B = \phi$ (vi) $(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$

Let us state some properties of set operations.

1.4 Properties of set operations

For any three sets A, B and C, the following hold.

(i) Commutative property

(a) $A \cup B = B \cup A$ (set union is commutative)

(b) $A \cap B = B \cap A$ (set intersection is commutative)

(ii) Associative property

(a) $A \cup (B \cup C) = (A \cup B) \cup C$ (set union is associative)

(b) $A \cap (B \cap C) = (A \cap B) \cap C$ (set intersection is associative)

(iii) Distributive property

(a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (intersection distributes over union)

(b) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (union distributes over intersection)

Mostly we shall verify these properties with the given sets. Instead of verifying the above properties with examples, it is always better to give a mathematical proof. But this is beyond the scope of this book. However, to understand and appreciate a rigorous mathematical proof, let us take one property and give the proof.

(i) Commutative property of union

In this part we want to prove that for any two sets A and B, the sets $A \cup B$ and $B \cup A$ are equal. Our definition of equality of sets says that two sets are equal only if they contain same elements.

First we shall show that every element of $A \cup B$, is also an element of $B \cup A$.
Let $z \in A \cup B$ be an arbitrary element. Then by the definition of union of A and B we have $z \in A$ or $z \in B$. That is,

$$\begin{aligned} \text{for every } z \in A \cup B &\Longrightarrow z \in A \text{ or } z \in B \\ &\Longrightarrow z \in B \text{ or } z \in A \\ &\Longrightarrow z \in B \cup A \text{ by the definition of } B \cup A. \qquad (1) \end{aligned}$$

Since (1) is true for every $z \in A \cup B$, the above work shows that every element of $A \cup B$ is also is an element of $B \cup A$. Hence, by the definition of a subset, we have $(A \cup B) \subseteq (B \cup A)$.

Next, we consider an arbitrary $y \in B \cup A$ and show that this y is also an element of $A \cup B$.

$$\begin{aligned} \text{Now, for every } y \in B \cup A &\Longrightarrow y \in B \text{ or } y \in A \\ &\Longrightarrow y \in A \text{ or } y \in B \\ &\Longrightarrow y \in A \cup B \text{ by the definition of } A \cup B. \qquad (2) \end{aligned}$$

Since (2) is true for every $y \in B \cup A$, the above work shows that every element of $B \cup A$ is also an element of $A \cup B$. Hence, by the definition of a subset, we have $(B \cup A) \subseteq (A \cup B)$.

So, we have shown that $(A \cup B) \subseteq (B \cup A)$ and $(B \cup A) \subseteq (A \cup B)$. This can happen only when $(A \cup B) = (B \cup A)$. One could follow above steps to prove other properties listed above by exactly the same method.

About proofs in Mathematics

In mathematics, a statement is called a **true statement** if it is always true. If a statement is not true even in one instance, then the statement is said to be a **false statement**. For example, let us consider a few statements:

(i) Any positive odd integer is a prime number

(ii) Sum of all angles in a triangle is 180°

(iii) Every prime number is an odd integer

(iv) For any two sets A and B, $A \setminus B = B \setminus A$

Now, the statement (i) is false, though very many odd positive integers are prime, because integers like $9, 15, 21, 45$ etc. are positive and odd but not prime.

The statement (ii) is a true statement because no matter which triangle you consider, the sum of its angles equals 180°.

The statement (iii) is false, because 2 is a prime number but it is an even integer. In fact, the statement (iii) is true for every prime number except for 2. So, if we want to prove a statement we have to prove that it is true for all instances. If we want to disprove a statement it is enough to give an example of one instance, where it is false.

The statement (iv) is false. Let us analyze this statement. Basically, when we form $A \setminus B$ we are removing all elements of B from A. Similarly, for $B \setminus A$. So it is highly possible that the above statement is false. Indeed, let us consider a case where $A = \{2, 5, 8\}$ and $B = \{5, 7, -1\}$. In this case, $A \setminus B = \{2, 8\}$ and $B \setminus A = \{7, -1\}$ and we have $A \setminus B \neq B \setminus A$. Hence the statement given in (iv) is false.

Example 1.1

For the given sets $A = \{-10, 0, 1, 9, 2, 4, 5\}$ and $B = \{-1, -2, 5, 6, 2, 3, 4\}$,

verify that (i) set union is commutative. Also verify it by using Venn diagram.

(ii) set intersection is commutative. Also verify it by using Venn diagram.

Solution

(i) Now, $A \cup B = \{-10, 0, 1, 9, 2, 4, 5\} \cup \{-1, -2, 5, 6, 2, 3, 4\}$

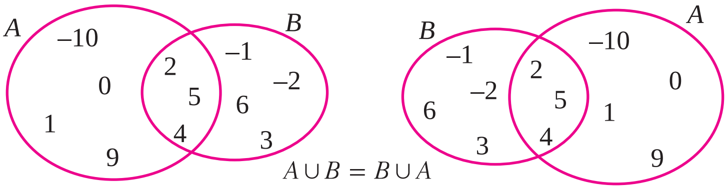
$$= \{-10, -2, -1, 0, 1, 2, 3, 4, 5, 6, 9\} \quad (1)$$

Also, $B \cup A = \{-1, -2, 5, 6, 2, 3, 4\} \cup \{-10, 0, 1, 9, 2, 4, 5\}$

$$= \{-10, -2, -1, 0, 1, 2, 3, 4, 5, 6, 9\} \quad (2)$$

Thus, from (1) and (2) we have verified that $A \cup B = B \cup A$.

By Venn diagram, we have



$A \cup B = B \cup A$

Fig. 1.7

Hence, it is verified that set union is commutative.

(ii) Let us verify that intersection is commutative.

Now, $A \cap B = \{-10, 0, 1, 9, 2, 4, 5\} \cap \{-1, -2, 5, 6, 2, 3, 4\}$

$= \{2, 4, 5\}$. (1)

Also, $B \cap A = \{-1, -2, 5, 6, 2, 3, 4\} \cap \{-10, 0, 1, 9, 2, 4, 5\}$

$= \{2, 4, 5\}$. (2)

From (1) and (2), we have $A \cap B = B \cap A$ for the given sets A and B.

By Venn diagram, we have

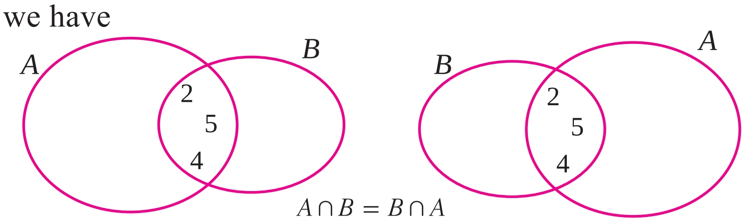


Fig. 1.8

Hence, it is verified.

Example 1.2

Given, $A = \{1, 2, 3, 4, 5\}$, $B = \{3, 4, 5, 6\}$ and $C = \{5, 6, 7, 8\}$, show that

(i) $A \cup (B \cup C) = (A \cup B) \cup C$. (ii) Verify (i) using Venn diagram.

Solution

(i) Now, $B \cup C = \{3, 4, 5, 6\} \cup \{5, 6, 7, 8\} = \{3, 4, 5, 6, 7, 8\}$

$\therefore$ $A \cup (B \cup C) = \{1, 2, 3, 4, 5\} \cup \{3, 4, 5, 6, 7, 8\} = \{1, 2, 3, 4, 5, 6, 7, 8\}$ (1)

Now, $A \cup B = \{1, 2, 3, 4, 5\} \cup \{3, 4, 5, 6\} = \{1, 2, 3, 4, 5, 6\}$

$\therefore$ $(A \cup B) \cup C = \{1, 2, 3, 4, 5, 6\} \cup \{5, 6, 7, 8\} = \{1, 2, 3, 4, 5, 6, 7, 8\}$ (2)

From (1) and (2), we have $A \cup (B \cup C) = (A \cup B) \cup C$.

(ii) Using Venn diagram, we have

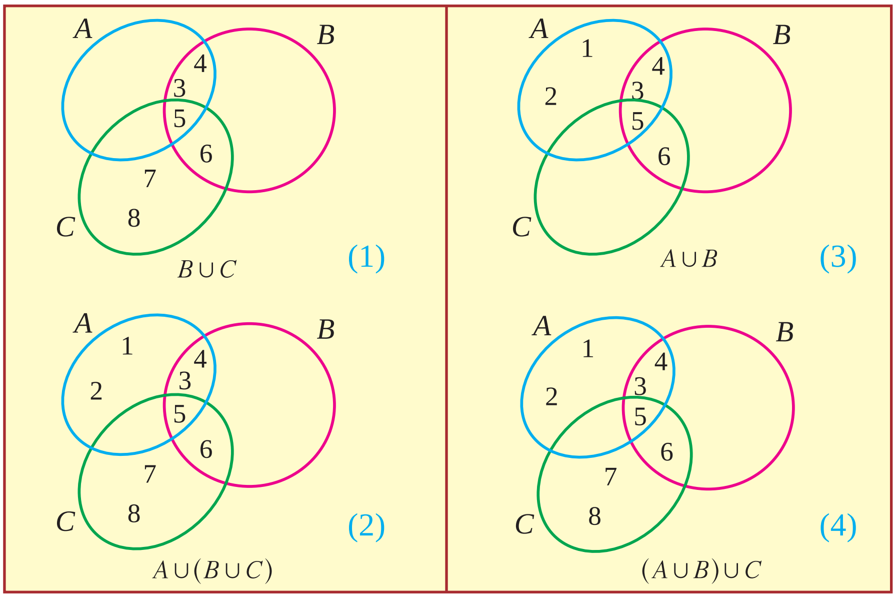


Fig. 1.9

Thus, from (2) and (4), we have verified that the set union is associative.

Example 1.3

Let $A = \{a,b,c,d\}$, $B = \{a,c,e\}$ and $C = \{a,e\}$.

(i) Show that $A \cap (B \cap C) = (A \cap B) \cap C$. (ii) Verify (i) using Venn diagram.

Solution

(i) We are given $A = \{a,b,c,d\}$, $B = \{a,c,e\}$ and $C = \{a,e\}$.

We need to show $A \cap (B \cap C) = (A \cap B) \cap C$. So, we first consider $A \cap (B \cap C)$.

Now, $B \cap C = \{a,c,e\} \cap \{a,e\} = \{a,e\}$; thus,

$$A \cap (B \cap C) = \{a,b,c,d\} \cap \{a,e\} = \{a\}. \quad (1)$$

Next, we shall find $A \cap B = \{a,b,c,d.\} \cap \{a,c,e\} = \{a,c\}$. Hence

$$(A \cap B) \cap C = \{a,c\} \cap \{a,e\} = \{a\} \quad (2)$$

Now (1) and (2) give the desired result.

(ii) Using Venn diagram, we have

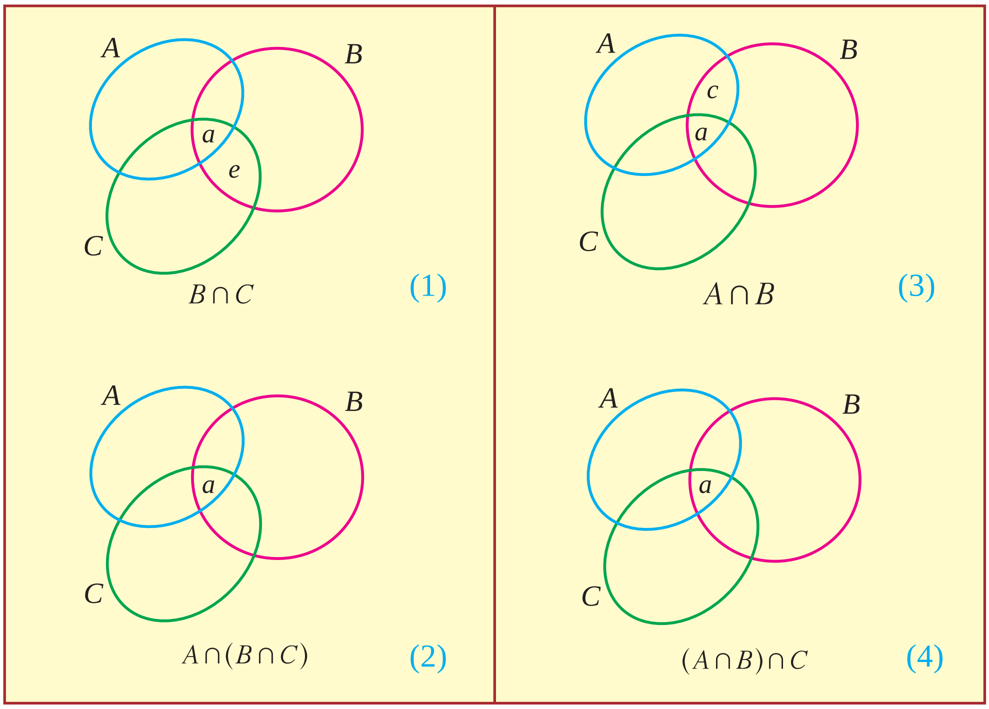


Fig. 1.10

Thus, from (2) and (4) , it is verified that $A \cap (B \cap C) = (A \cap B) \cap C$

Example 1.4

Given $A = \{a,b,c,d,e\}$, $B = \{a,e,i,o,u\}$ and $C = \{c,d,e,u\}$.

(i) Show that $A \setminus (B \setminus C) \neq (A \setminus B) \setminus C$. (ii) Verify (i) using Venn diagram.

Solution

(i) First let us find $A\setminus(B\setminus C)$. To do so, consider

$$(B\setminus C) = \{a,e,i,o,u\}\setminus\{c,d,e,u\} = \{a,i,o\}.$$

Thus, $A\setminus(B\setminus C) = \{a,b,c,d,e\}\setminus\{a,i,o\} = \{b,c,d,e\}$. (1)

Next, we find $(A\setminus B)\setminus C$.

$$A\setminus B = \{a,b,c,d,e\}\setminus\{a,e,i,o,u\} = \{b,c,d\}.$$

Hence, $(A\setminus B)\setminus C = \{b,c,d\}\setminus\{c,d,e,u\} = \{b\}$. (2)

From (1) and (2) we see that $A\setminus(B\setminus C) \neq (A\setminus B)\setminus C$.

Thus, the set difference is not associative.

(ii) Using Venn diagram, we have

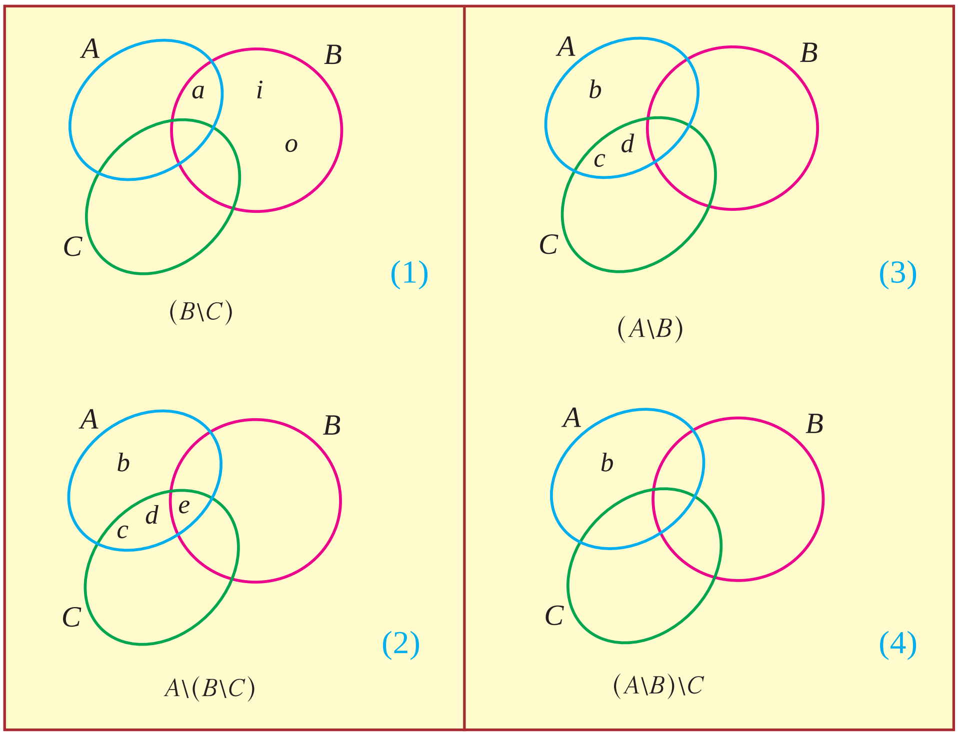


Fig. 1.11

From (2) and (4), it is verified that $A\setminus(B\setminus C) \neq (A\setminus B)\setminus C$.

Remarks

The set difference is not associative. However, if the sets A, B and C are mutually disjoint, then $A\setminus(B\setminus C) = (A\setminus B)\setminus C$. This is very easy to prove; so let us prove it. Since B and C are disjoint we have $B\setminus C = B$. Since A, B are disjoint we have $A\setminus B = A$. Thus, we have $A\setminus(B\setminus C) = A$. Again, $A\setminus B = A$ and A, C are disjoint and so we have $A\setminus C = A$. Hence, $(A\setminus B)\setminus C = A$. So we have $A\setminus(B\setminus C) = (A\setminus B)\setminus C$ as desired. Thus, for sets which are mutually disjoint, the set difference is associative.

Example 1.5

Let $A = \{0,1,2,3,4\}$, $B = \{1,-2,3,4,5,6\}$ and $C = \{2,4,6,7\}$.

(i) Show that $A\cup(B\cap C) = (A\cup B)\cap(A\cup C)$. (ii) Verify using Venn diagram.

Solution

(i) First, we find $A\cup(B\cap C)$.

Consider $B\cap C = \{1,-2,3,4,5,6\}\cap\{2,4,6,7\} = \{4,6\}$;

$$A\cup(B\cap C) = \{0,1,2,3,4\}\cup\{4,6\} = \{0,1,2,3,4,6\}. \quad (1)$$

Next, consider $A\cup B = \{0,1,2,3,4\}\cup\{1,-2,3,4,5,6\}$

$= \{-2,0,1,2,3,4,5,6\}$,

$A\cup C = \{0,1,2,3,4\}\cup\{2,4,6,7\} = \{0,1,2,3,4,6,7\}$.

Thus, $(A\cup B)\cap(A\cup C) = \{-2,0,1,2,3,4,5,6\}\cap\{0,1,2,3,4,6,7\}$

$= \{0,1,2,3,4,6\}. \quad (2)$

From (1) and (2), we get $A\cup(B\cap C) = (A\cup B)\cap(A\cup C)$.

(ii) Using Venn diagram, we have

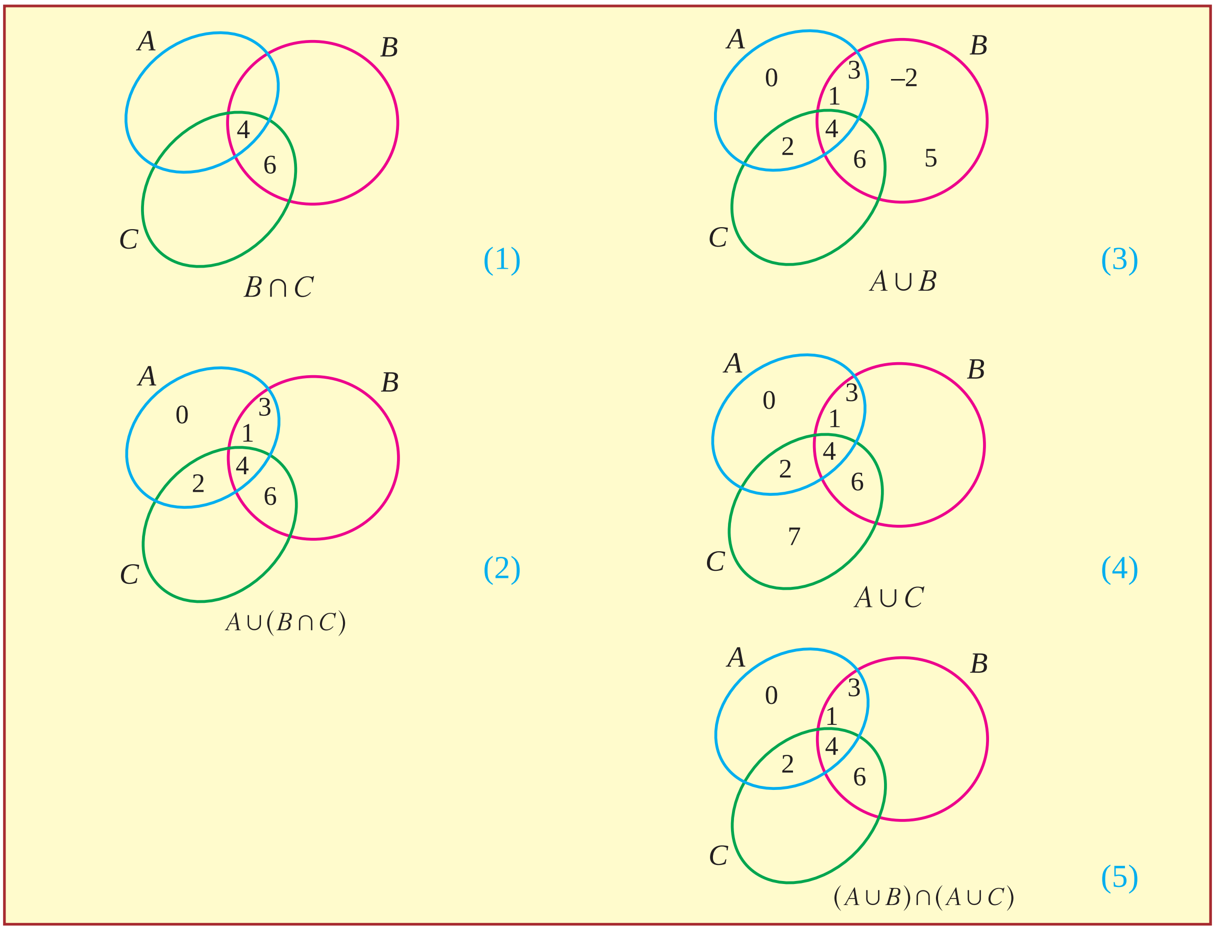


Fig. 1.12

From (2) and (5) it is verified that $A\cup(B\cap C) = (A\cup B)\cap(A\cup C)$

Example 1.6

For $A = \{x \mid -3 \leq x < 4, x \in \mathbb{R}\}, B = \{x \mid x < 5, x \in \mathbb{N}\}$ and

$C = \{-5, -3, -1, 0, 1, 3\}$, Show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Solution First note that the set A contains all the real numbers (not just integers) that are greater than or equal to -3 and less than 4.

On the other hand the set B contains all the positive integers that are less than 5. So,

$A = \{x \mid -3 \leq x < 4, x \in \mathbb{R}\}$; that is, A consists of all real numbers from – 3 upto 4 but 4 is not included.

Also, $B = \{x \mid x < 5, x \in \mathbb{N}\} = \{1, 2, 3, 4\}$. Now, we find

$$B \cup C = \{1, 2, 3, 4\} \cup \{-5, -3, -1, 0, 1, 3\}$$

$$= \{1, 2, 3, 4, -5, -3, -1, 0\}; \text{ thus}$$

$$A \cap (B \cup C) = A \cap \{1, 2, 3, 4, -5, -3, -1, 0\}$$

$$= \{-3, -1, 0, 1, 2, 3\}. \qquad (1)$$

Next, to find $(A \cap B) \cup (A \cap C)$, we consider

$$A \cap B = \{x \mid -3 \leq x < 4, x \in \mathbb{R}\} \cap \{1, 2, 3, 4\} = \{1, 2, 3\};$$

and $$A \cap C = \{x \mid -3 \leq x < 4, x \in \mathbb{R}\} \cap \{-5, -3, -1, 0, 1, 3\}$$

$$= \{-3, -1, 0, 1, 3\}.$$

Hence, $$(A \cap B) \cup (A \cap C) = \{1, 2, 3,\} \cup \{-3, -1, 0, 1, 3\}$$

$$= \{-3, -1, 0, 1, 2, 3\}. \qquad (2)$$

Now (1) and (2) imply $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Exercise 1.1

1. If $A \subset B$, then show that $A \cup B = B$ (use Venn diagram).
2. If $A \subset B$, then find $A \cap B$ and $A \setminus B$ (use Venn diagram).
3. Let $P = \{a, b, c\}, Q = \{g, h, x, y\}$ and $R = \{a, e, f, s\}$. Find the following:

   (i) $P \setminus R$ (ii) $Q \cap R$ (iii) $R \setminus (P \cap Q)$.
4. If $A = \{4, 6, 7, 8, 9\}$, $B = \{2, 4, 6\}$ and $C = \{1, 2, 3, 4, 5, 6\}$, then find

   (i) $A \cup (B \cap C)$ (ii) $A \cap (B \cup C)$ (iii) $A \setminus (C \setminus B)$
5. Given $A = \{a, x, y, r, s\}, B = \{1, 3, 5, 7, -10\}$, verify the commutative property of set union.

6. Verify the commutative property of set intersection for
$A = \{l, m, n, o, 2, 3, 4, 7\}$ and $B = \{2, 5, 3, -2, m, n, o, p\}$.

7. For $A = \{x \mid x$ is a prime factor of $42\}$, $B = \{x \mid 5 < x \leq 12, x \in \mathbb{N}\}$ and
$C = \{1, 4, 5, 6\}$, verify $A \cup (B \cup C) = (A \cup B) \cup C$.

8. Given $P = \{a, b, c, d, e\}$, $Q = \{a, e, i, o, u\}$ and $R = \{a, c, e, g\}$. Verify the associative property of set intersection.

9. For $A = \{5, 10, 15, 20\}$; $B = \{6, 10, 12, 18, 24\}$ and $C = \{7, 10, 12, 14, 21, 28\}$, verify whether $A \setminus (B \setminus C) = (A \setminus B) \setminus C$. Justify your answer.

10. Let $A = \{-5, -3, -2, -1\}$, $B = \{-2, -1, 0\}$, and $C = \{-6, -4, -2\}$. Find $A \setminus (B \setminus C)$ and $(A \setminus B) \setminus C$. What can we conclude about set difference operation?

11. For $A = \{-3, -1, 0, 4, 6, 8, 10\}$, $B = \{-1, -2, 3, 4, 5, 6\}$ and $C = \{-1, 2, 3, 4, 5, 7\}$, show that (i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
(iii) Verify (i) using Venn diagram (iv) Verify (ii) using Venn diagram.

1.5 De Morgan's laws

De Morgan's father (a British national) was in the service of East India Company, India. Augustus De Morgan (1806-1871) was born in Madurai, Tamilnadu, India. His family moved to England when he was seven months old. He had his education at Trinity college, Cambridge, England. De Morgan's laws relate the three basic set operations Union, Intersection and Complementation.

De Morgan's laws for set difference

For any three sets A, B and C, we have
(i) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$ (ii) $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.

De Morgan's laws for complementation

Let U be the universal set containing sets A and B. Then
(i) $(A \cup B)' = A' \cap B'$ (ii) $(A \cap B)' = A' \cup B'$.

Observe that proof of the laws for complementation follows from that of the set difference because for any set D, we have $D' = U \setminus D$. Again we shall not attempt to prove these; but we shall learn how to apply these laws in problem solving.

Example 1.7

Use Venn diagrams to verify $(A \cap B)' = A' \cup B'$.

Solution

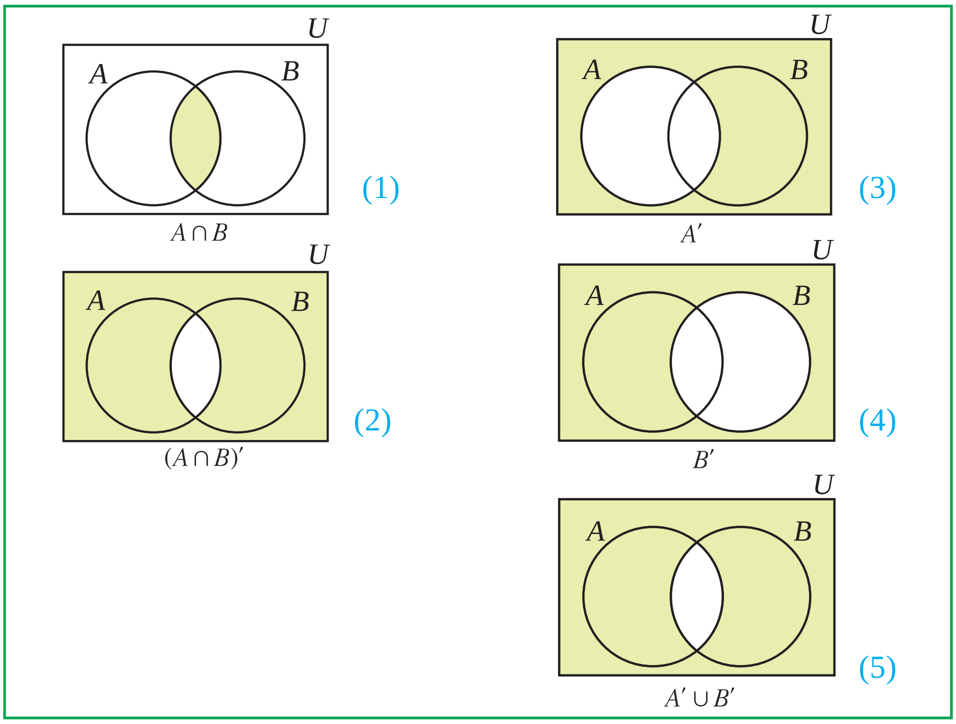


Fig. 1.13

From (2) and (5) it follows that $(A \cap B)' = A' \cup B'$.

Example 1.8

Use Venn diagrams to verify De Morgan's law for set difference

$$A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C).$$

Solution

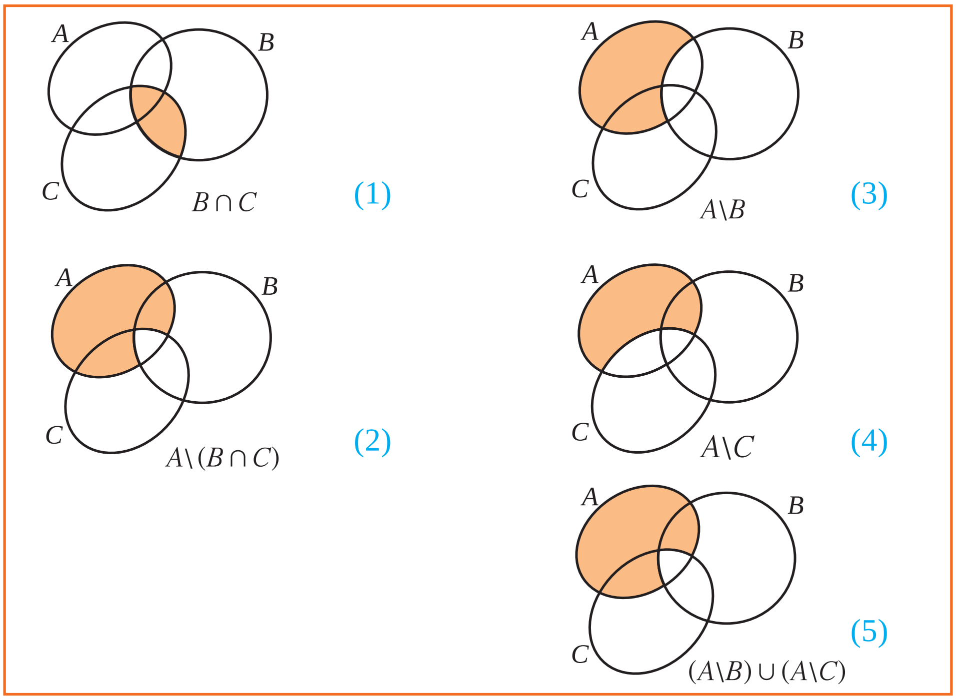


Fig. 1.14

From (2) and (5) we have $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.

Example 1.9

Let $U = \{-2, -1, 0, 1, 2, 3, \cdots, 10\}$, $A = \{-2, 2, 3, 4, 5\}$ and $B = \{1, 3, 5, 8, 9\}$.

Verify De Morgan's laws of complementation.

Solution First we shall verify $(A \cup B)' = A' \cap B'$. To do this we consider

$$A \cup B = \{-2, 2, 3, 4, 5\} \cup \{1, 3, 5, 8, 9\} = \{-2, 1, 2, 3, 4, 5, 8, 9\};$$

which implies

$$(A \cup B)' = U \setminus \{-2, 1, 2, 3, 4, 5, 8, 9\} = \{-1, 0, 6, 7, 10\}. \quad (1)$$

Next, we find
$$A' = U \setminus A = \{-1, 0, 1, 6, 7, 8, 9, 10\}$$
$$B' = U \setminus B = \{-2, -1, 0, 2, 4, 6, 7, 10\}.$$

Thus, we have
$$A' \cap B' = \{-1, 0, 1, 6, 7, 8, 9, 10\} \cap \{-2, -1, 0, 2, 4, 6, 7, 10\}$$
$$= \{-1, 0, 6, 7, 10\}. \quad (2)$$

From (1) and (2) it follows that $(A \cup B)' = A' \cap B'$.

Similarly, one can verify $(A \cap B)' = A' \cup B'$ for the given sets above. We leave the details as an exercise.

Example 1.10

Let $A = \{a, b, c, d, e, f, g, x, y, z\}$, $B = \{1, 2, c, d, e\}$ and $C = \{d, e, f, g, 2, y\}$.

Verify $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.

Solution First, we find
$$B \cup C = \{1, 2, c, d, e\} \cup \{d, e, f, g, 2, y\}$$
$$= \{1, 2, c, d, e, f, g, y\}.$$

Then
$$A \setminus (B \cup C) = \{a, b, c, d, e, f, g, x, y, z\} \setminus \{1, 2, c, d, e, f, g, y\}$$
$$= \{a, b, x, z\}. \quad (1)$$

Next, we have $A \setminus B = \{a, b, f, g, x, y, z\}$ and $A \setminus C = \{a, b, c, x, z\}$

and so
$$(A \setminus B) \cap (A \setminus C) = \{a, b, x, z\}. \quad (2)$$

Hence, from (1) and (2) it follows that $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.

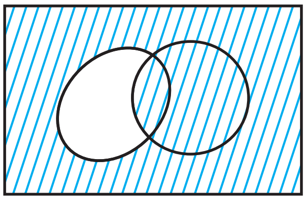
Exercise 1.2

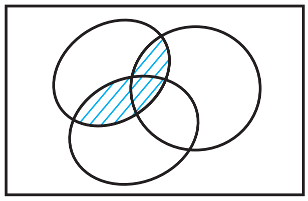
1. Represent the following using Venn diagrams

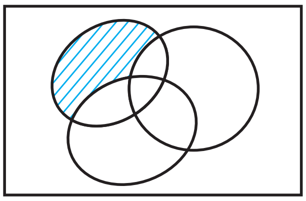
   (i) $U = \{5, 6, 7, 8, ...13\}$, $A = \{5, 8, 10, 11\}$, and $B = \{5, 6, 7, 9, 10\}$

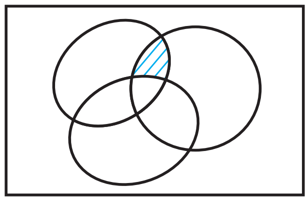
   (ii) $U = \{a, b, c, d, e, f, g, h\}$, $M = \{b, d, f, g\}$, and $N = \{a, b, d, e, g\}$

2. Write a description of each shaded area. Use symbols U, $A, B, C, \cup, \cap, '$ and $\setminus$ as necessary.

(i) 

(ii) 

(iii) 

(iv) 

3. Draw Venn diagram of three sets A, B and C illustrating the following:

(i) $A \cap B \cap C$ (ii) A and B are disjoint but both are subsets of C

(iii) $A \cap (B \setminus C)$ (iv) $(B \cup C) \setminus A$ (v) $A \cup (B \cap C)$

(vi) $C \cap (B \setminus A)$ (vii) $C \cap (B \cup A)$

4. Use Venn diagram to verify $(A \cap B) \cup (A \setminus B) = A$.

5. Let $U = \{4, 8, 12, 16, 20, 24, 28\}$, $A = \{8, 16, 24\}$ and $B = \{4, 16, 20, 28\}$. Find $(A \cup B)'$ and $(A \cap B)'$.

6. Given that $U = \{a, b, c, d, e, f, g, h\}$, $A = \{a, b, f, g\}$, and $B = \{a, b, c\}$, verify De Morgan's laws of complementation.

7. Verify De Morgan's laws for set difference using the sets given below:

$A = \{1, 3, 5, 7, 9, 11, 13, 15\}$, $B = \{1, 2, 5, 7\}$ and $C = \{3, 9, 10, 12, 13\}$.

8. Let $A = \{10, 15, 20, 25, 30, 35, 40, 45, 50\}$, $B = \{1, 5, 10, 15, 20, 30\}$ and $C = \{7, 8, 15, 20, 35, 45, 48\}$. Verify $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.

9. Using Venn diagram, verify whether the following are true:

(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(iii) $(A \cup B)' = A' \cap B'$ (iv) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$

1.6 Cardinality of sets

In class IX, we have learnt to solve problems involving two sets, using the formula $n(A\cup B) = n(A) + n(B) - n(A\cap B)$. This formula helps us in calculating the cardinality of the set $A\cup B$ when the cardinalities of A, B and $A\cap B$ are known. Suppose we have three sets A, B and C and we want to find the cardinality of $A\cup B\cup C$, what will be the corresponding formula? The formula in this case is given by

$$n(A\cup B\cup C) = n(A) + n(B) + n(C) - n(A\cap B) - n(B\cap C) - n(A\cap C) + n(A\cap B\cap C).$$

Following example illustrates the usage of the above formula.

Example 1.11

In a group of students, 65 play foot ball, 45 play hockey, 42 play cricket, 20 play foot ball and hockey, 25 play foot ball and cricket, 15 play hockey and cricket and 8 play all the three games. Find the number of students in the group.
(Assume that each student in the group plays atleast one game.)

Solution Let F, H and C represent the set of students who play foot ball, hockey and cricket respectively. Then $n(F) = 65$, $n(H) = 45$, and $n(C) = 42$.

Also, $n(F\cap H) = 20$, $n(F\cap C) = 25$, $n(H\cap C) = 15$ and $n(F\cap H\cap C) = 8$.

We want to find the number of students in the whole group; that is $n(F\cup H\cup C)$.

By the formula, we have

$$\begin{aligned} n(F\cup H\cup C) &= n(F) + n(H) + n(C) - n(F\cap H) \\ &\qquad - n(H\cap C) - n(F\cap C) + n(F\cap H\cap C) \\ &= 65 + 45 + 42 - 20 - 25 - 15 + 8 = 100. \end{aligned}$$

Hence, the number of students in the group = 100.

Alternate method

The same problem can also be solved using Venn diagram. Nowadays, it is possible to solve some of the problems that we come across in daily life using Venn diagrams and logic. The Venn diagram will have three intersecting sets, each representing a game. Look at the diagram and try to find the number of players in the group by working carefully through the statements and fill in as you go along.

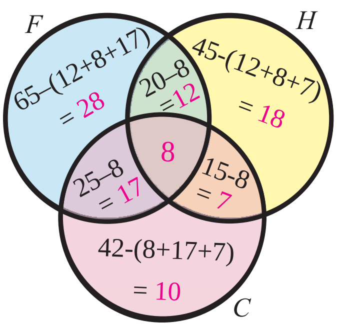


Fig. 1.15

Number of students in the group

= 28 + 12 + 18 + 7 + 10 + 17 + 8 = 100.

Example 1.12

In a survey of university students, 64 had taken mathematics course, 94 had taken computer science course, 58 had taken physics course, 28 had taken mathematics and physics, 26 had taken mathematics and computer science, 22 had taken computer science and physics course, and 14 had taken all the three courses. Find the number of students who were surveyed. Find how many had taken one course only.

Solution Let us represent the given data in a Venn diagram.

Let M, C, P represent sets of students who had taken mathematics, computer science and physics respectively. The given details are filled in the Venn diagram

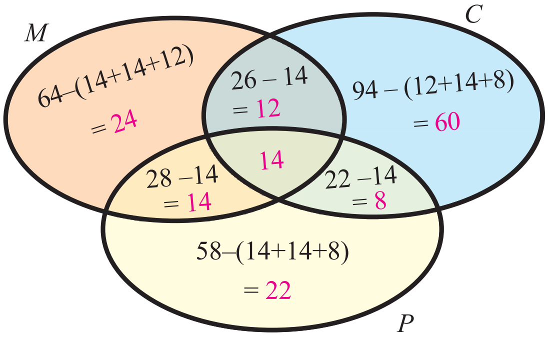


Fig. 1.16

$n(M \cap C \cap P) = 14$

$n(M \cap C \cap P') = 26 - 14 = 12$

$n(M \cap P \cap C') = 28 - 14 = 14$

$n(C \cap P \cap M') = 22 - 14 = 8$

Number of students surveyed

$= 24 + 12 + 60 + 8 + 22 + 14 + 14 = 154$

The number of students who had taken only mathematics = 64–(14+14+12) = 24

The number of students who had taken only computer science = 94 – (12+14+8) = 60

The number of students who had taken only physics = 58 – (14+14+8) = 22

The number of students who had taken one course only = 24+60+22 = 106.

Example 1.13

A radio station surveyed 190 students to determine the types of music they liked. The survey revealed that 114 liked rock music, 50 liked folk music, and 41 liked classical music, 14 liked rock music and folk music, 15 liked rock music and classical music, 11 liked classical music and folk music. 5 liked all the three types of music.

Find (i) how many did not like any of the 3 types?

(ii) how many liked any two types only?

(iii) how many liked folk music but not rock music?

Solution Let R, F and C represent the sets of students who liked rock music, folk music and classical music respectively. Let us fill in the given details in the Venn diagram. Thus, we have

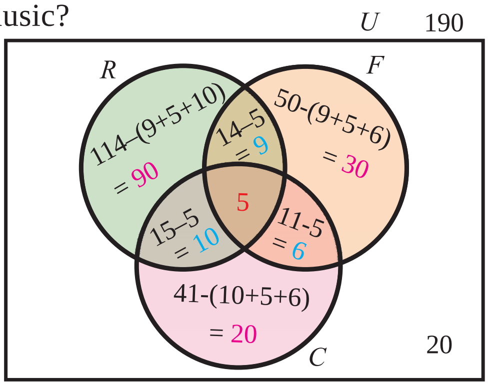


Fig. 1.17

$n(R \cap F \cap C') = 14 - 5 = 9$

$n(R \cap C \cap F') = 15 - 5 = 10$

$n(F \cap C \cap R') = 11 - 5 = 6.$

From the Venn diagram, the number of students who liked any one of the three types of music equals 90 + 9 + 30 + 6 + 20 + 10 + 5 = 170.

Number of students surveyed = 190.

Number of students who did not like any of the three types = $190-170=20$.

Number of students who liked any two types only = 9 + 6 + 10 = 25.

Number of students who liked folk music but not rock music = 30 + 6 = 36.

Exercise 1.3

1. If A and B are two sets and U is the universal set such that $n(U)=700$, $n(A)=200$, $n(B)=300$ and $n(A\cap B)=100$, find $n(A'\cap B')$.

2. Given $n(A)=285$, $n(B)=195$, $n(U)=500$, $n(A\cup B)=410$, find $n(A'\cup B')$.

3. For any three sets A, B and C if $n(A)=17$ $n(B)=17$, $n(C)=17$, $n(A\cap B)=7$ $n(B\cap C)=6$, $n(A\cap C)=5$ and $n(A\cap B\cap C)=2$, find $n(A\cup B\cup C)$.

4. Verify $n(A\cup B\cup C)=n(A)+n(B)+n(C)-n(A\cap B)-$ $n(B\cap C)-n(A\cap C)+n(A\cap B\cap C)$ for the sets given below:
   (i) $A=\{4,5,6\}$, $B=\{5,6,7,8\}$ and $C=\{6,7,8,9\}$
   (ii) $A=\{a,b,c,d,e\}$, $B=\{x,y,z\}$ and $C=\{a,e,x\}$.

5. In a college, 60 students enrolled in chemistry, 40 in physics, 30 in biology, 15 in chemistry and physics, 10 in physics and biology, 5 in biology and chemistry. No one enrolled in all the three. Find how many are enrolled in at least one of the subjects.

6. In a town 85% of the people speak Tamil, 40% speak English and 20% speak Hindi. Also, 32% speak English and Tamil, 13% speak Tamil and Hindi and 10% speak English and Hindi, find the percentage of people who can speak all the three languages.

7. An advertising agency finds that, of its 170 clients, 115 use Television, 110 use Radio and 130 use Magazines. Also, 85 use Television and Magazines, 75 use Television and Radio, 95 use Radio and Magazines, 70 use all the three. Draw Venn diagram to represent these data. Find
   (i) how many use only Radio? (ii) how many use only Television?
   (iii) how many use Television and magazine but not radio?

8. In a school of 4000 students, 2000 know French, 3000 know Tamil and 500 know Hindi, 1500 know French and Tamil, 300 know French and Hindi, 200 know Tamil and Hindi and 50 know all the three languages.
   (i) How many do not know any of the three languages?
   (ii) How many know at least one language?
   (iii) How many know only two languages?

9. In a village of 120 families, 93 families use firewood for cooking, 63 families use kerosene, 45 families use cooking gas, 45 families use firewood and kerosene, 24 families use kerosene and cooking gas, 27 families use cooking gas and firewood. Find how many use firewood, kerosene and cooking gas.

1.7 Relations

In the previous section, we have seen the concept of Set. We have also seen how to create new sets from the given sets by taking union, intersection and complementation. Here we shall see yet another way of creating a new set from the given two sets *A* and *B*. This new set is important in defining other important concepts of mathematics "relation, function".

Given two non empty sets *A* and *B*, we can form a new set $A \times B$, read as '*A* cross *B*', called the cartesian product of A with B. It is defined as

$$A \times B = \{(a,b) \mid a \in A \text{ and } b \in B\}.$$

Similarly, the set B cross A is defined as

$$B \times A = \{(b,a) \mid b \in B \text{ and } a \in A\}.$$

Note

(i) The order in the pair $(a,\ b)$ is important. That is, $(a,\ b) \neq (b,\ a)$ if $a \neq b$.

(ii) It is possible that the sets A and B are equal in the cartesian product $A \times B$.

Let us look at an example.

Suppose that a cell phone store sells three different types of cell phones and we call them C_1, C_2, C_3. Let us also suppose that the price of C_1 is ₹ 1200, price of C_2 is ₹ 2500 and price of C_3 is ₹ 2500.

We take $A = \{ C_1, C_2, C_3 \}$ and $B = \{ 1200, 2500 \}$.

In this case, $A \times B = \{(C_1, 1200), (C_1, 2500), (C_2, 1200), (C_2, 2500), (C_3, 1200), (C_3, 2500)\}$

but $B \times A = \{(1200, C_1), (2500, C_1), (1200, C_2), (2500, C_2,),(1200, C_3), (2500, C_3)$.

It is easy to see that $A \times B \neq B \times A$ if $A \neq B$.

Let us consider a subset $F = \{(C_1, 1200), (C_2, 2500), (C_3, 2500)\}$ of $A \times B$.

Every first component in the above ordered pairs is associated with a unique element. That is no element in the first place is paired with more than one element in the second place.

For every element in F, basically the second component indicates the price of the first component. Next, consider a subset $E = \{(1200, C_1), (2500, C_2), (2500, C_3)\}$ of $B \times A$

Here, the first component 2500 is associated with two different elements C_2 and C_3.

Definition

Let A and B be any two non empty sets. A relation R from A to B is a non-empty subset of $A \times B$. That is, $R \subseteq A \times B$.

Domain of $R = \{x \in A \mid (x,y) \in R \text{ for some } y \in B\}$

Range of $R = \{y \in B \mid (x,y) \in R \text{ for some } x \in A\}$.

1.8 Functions

Peter Dirichlet

(1805-1859)
Germany

Dirichlet made major contributions in the fields of number theory, analysis and mechanics.

In 1837 he introduced the modern concept of a function with notation $y = f(x)$. He also formulated the well known Pigeonhole principle.

Let A and B be any two non empty sets. A function from A to B is a relation

$f \subseteq A \times B$ such that the following hold:

(i) Domain of f is A.

(ii) For each $x \in A$, there is only one $y \in B$ such that $(x, y) \in f$.

Note that a function from A to B is a special kind of relation that satisfies (i) and (ii). A function is also called as a mapping or a transformation.

A function from A to B is denoted by $f{:}A \to B$, and if $(x,y) \in f$, then we write $y = f(x)$.

We can reformulate the definition of a function without using the idea of relation as follows: In fact, most of the time this formulation is used as a working definition of a function,

Definition

Let A and B be any two non empty sets. A function f from A to B is a rule of correspondence that assigns each element $x \in A$ to a unique element $y \in B$. We denote $y = f(x)$ to mean y is a function of x.

The set A is called the **domain** of the function and set B is called the **co-domain** of the function. Also, y is called **the image** of x under f and x is called **a preimage** of y. The set of all images of elements of A under f is called the **range** of f. Note that the range of a function is a subset of its co-domain.

This modern definition of a function, given above, was given by Nikolai Labachevsky and **Peter Dirichlet** independently around 1837. Prior to this, there was no clear definition of a function.

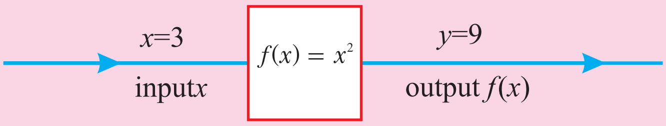
In the example we considered in section 1.7, prior to the above definitions, the set

$F = \{(C_1, 1200), (C_2, 2500), (C_3, 2500)\}$ represents a function; because $F \subseteq A \times B$ is a relation satisfying conditions (i) and (ii) given above.

But $E = \{(1200, C_1), (2500, C_2), (2500, C_3)\}$ does not represent a function, because condition (ii) given above is not satisfied as $(2500, C_2), (2500, C_3) \in E$.

Remarks

(i) A function f may be thought of as a machine which yields a unique output y for every input value of x.



(ii) In defining a function we need a domain, co-domain and a rule that assigns each element of the domain to a unique element in the co-domain.

Example 1.14

Let $A = \{1, 2, 3, 4\}$ and $B = \{-1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 12\}$.

Let $R = \{(1, 3), (2, 6), (3, 10), (4, 9)\} \subseteq A \times B$ be a relation. Show that R is a function and find its domain, co-domain and the range of R.

Solution The domain of $R = \{1, 2, 3, 4\} = A$.

Also, for each $x \in A$ there is only one $y \in B$ such that $y = R(x)$.

So, given R is a function. The co-domain is obviously B. Since

$R(1) = 3, R(2) = 6, R(3) = 10$ and $R(4) = 9$, the range of R is given by $\{3, 6, 10, 9\}$.

Example 1.15

Does each of the following arrow diagrams represent a function? Explain.

(i)

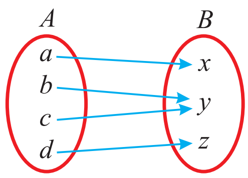


Fig. 1.18

(ii)

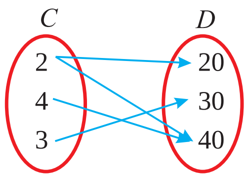


Fig. 1.19

Solution In arrow diagram (i), every element in A has a unique image. Hence it is a function. In arrow diagram (ii), the element 2 in C has two images namely 20 and 40. Hence, it is not a function.

Example 1.16

Let $X = \{ 1, 2, 3, 4 \}$. Examine whether each of the relations given below is a function from X to X or not. Explain.

(i) $f = \{ (2, 3), (1, 4), (2, 1), (3, 2), (4, 4) \}$

(ii) $g = \{ (3, 1), (4, 2), (2, 1) \}$ (iii) $h = \{ (2, 1), (3, 4), (1, 4), (4, 3) \}$

Solution

(i) Now, f = { (2, 3), (1, 4), (2, 1), (3, 2), (4, 4) }

f is not a function because 2 is associated with two different elements 3 and 1.

(ii) The relation g = { (3, 1), (4, 2), (2, 1)} is not a function because the element 1 does not have a image. That is, domain of $g = \{2, 3, 4\} \neq X$.

(iii) Next, we consider h = { (2, 1), (3, 4), (1, 4), (4, 3) }.

Each element in X is associated with a unique element in X.

Thus, h is a function.

Example 1.17

Which of the following relations are functions from A = { 1, 4, 9, 16 } to

B = { –1, 2, –3, –4, 5, 6 }? In case of a function, write down its range.

(i) f_1 = { (1, –1), (4, 2), (9, –3), (16, –4) }

(ii) f_2 = { (1, –4), (1, –1), (9, –3), (16, 2) }

(iii) f_3 = { (4, 2), (1, 2), (9, 2), (16, 2) }

(iv) f_4 = { (1, 2), (4, 5), (9, –4), (16, 5) }

Solution (i) We have f_1 = { (1, –1), (4, 2), (9, – 3), (16,– 4) }.

Each element in A is associated with a unique element in B.

Thus, f_1 is a function.

Range of f_1 is $\{-1, 2, -3, -4\}$.

(ii) Here, we have f_2 = { (1, – 4), (1, –1), (9, – 3), (16, 2) }.

f_2 is not a function because 1 is associated with two different image elements -4 and -1. Also, note that f_2 is not a function since 4 has no image.

(iii) Consider f_3 = { (4, 2), (1, 2), (9, 2), (16, 2) }.

Each element in A is associated with a unique element in B.

Thus, f_3 is a function.

Range of f_3 = { 2 }.

(iv) We have f_4 = { (1, 2), (4, 5), (9, – 4), (16, 5) }.

Each element in A is associated with a unique element in B.

Hence, f_4 is a function.

Range of f_4 = { 2, 5, – 4}.

Example 1.18

Let $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$, where $x \in \mathbb{R}$. Does the relation $\{(x,y) \mid y = |x|, x \in \mathbb{R}\}$ define a function? Find its range.

Solution For every value of x, there exists a unique value $y = |x|$. Therefore, the given relation defines a function.

The domain of the function is the set $\mathbb{R}$ of all real numbers.

Since $|x|$ is always either zero or positive for every real number x, and every positive real number can be obtained as an image under this function, the range will be the set of non-negative real numbers (either positive or zero).

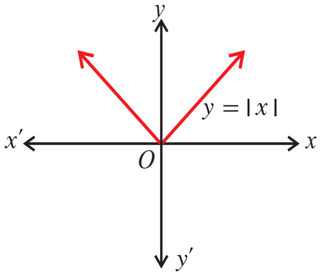


Fig. 1.20

Remarks

The function $y = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$, where $x \in \mathbb{R}$, is known as **modulus** or **absolute value** function.

Thus, for example, $|-8| = -(-8) = 8$ and also $|8| = 8$.

1.8.1 Representation of functions

A function may be represented by

(i) a set of ordered pairs, (ii) a table, (iii) an arrow diagram, (iv) a graph

Let $f : A \to B$ be a function.

(i) The set $f = \{(x, y) : y = f(x), x \in A\}$ of all ordered pairs represents the function.

(ii) The values of x and the values of their respective images under f can be given in the form of a table.

(iii) An arrow diagram indicates the elements of the domain of f and their respective images by means of arrows.

(iv) The ordered pairs in the collection $f = \{(x, y) : y = f(x), x \in A\}$ are plotted as points in the x-y plane. The graph of f is the totality of all such points.

Let us illustrate the representation of functions in different forms through some examples.

For many functions we can obtain its graph. But not every graph will represent a function. Following test helps us in determining if the given graph is a function or not.

1.8.2 Vertical line test

A graph represents a function only if every vertical line intersects the graph in at most one point.

Note

It is possible that some vertical lines may not intersect the graph, which is alright. If there is even one vertical line that meets the graph in more than one point, then that graph cannot represent a function, because in this case, we shall have at least two y-values for the same x-value. For example, the graph of $y^2 = x$ is not a function.

Example 1.19

Use the vertical line test to determine which of the following graphs represent a function.

(i)

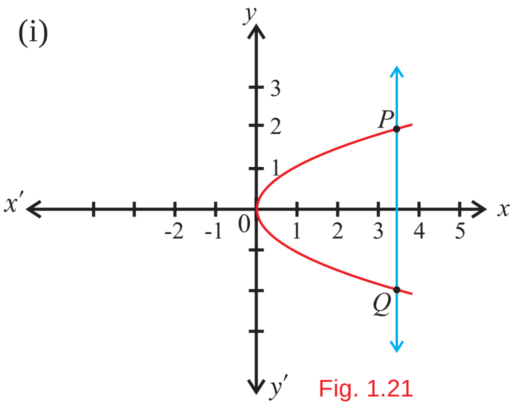


Fig. 1.21

(ii)

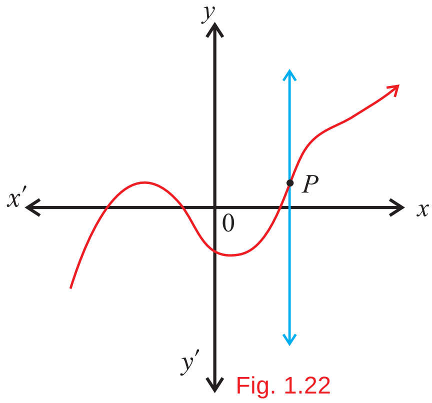


Fig. 1.22

(iii)

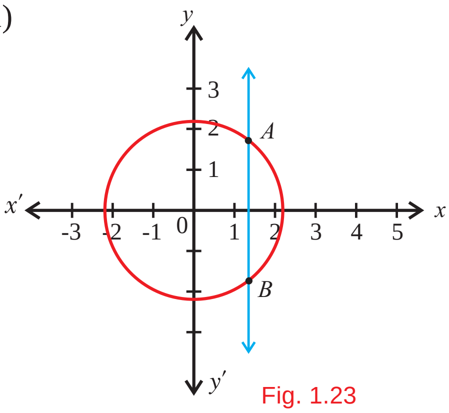


Fig. 1.23

(iv)

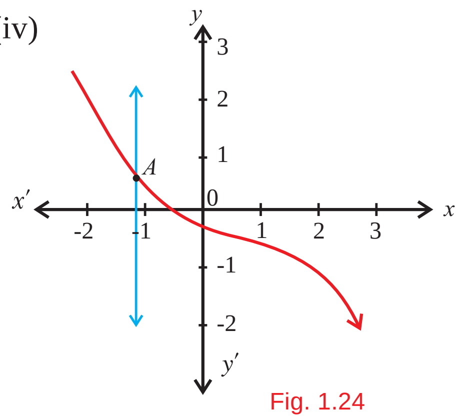


Fig. 1.24

Solution

(i) The given graph does not represent a function as a vertical line cuts the graph at two points P and Q.

(ii) The given graph represents a function as any vertical line will intersect the graph at most one point P.

(iii) The given graph does not represent a function as a vertical line cuts the graph at two points A and B.

(iv) The given graph represents a function as the graph satisfies the vertical line test.

Example 1.20

Let $A = \{ 0, 1, 2, 3 \}$ and $B = \{ 1, 3, 5, 7, 9 \}$ be two sets. Let $f : A \rightarrow B$ be a function given by $f(x) = 2x + 1$. Represent this function as (i) a set of ordered pairs (ii) a table (iii) an arrow diagram and (iv) a graph.

Solution A = $\{ 0, 1, 2, 3 \}$, $B = \{ 1, 3, 5, 7, 9 \}$, $f(x) = 2x + 1$

$f(0) = 2(0) + 1 = 1$, $f(1) = 2(1)+1 = 3$, $f(2) = 2(2) + 1 = 5$, $f(3) = 2(3) + 1 = 7$

(i) Set of ordered pairs

The given function f can be represented as a set of ordered pairs as

$$f = \{ (0, 1), (1, 3), (2, 5), (3, 7) \}$$

(ii) Table form

Let us represent f using a table as shown below.

x	0	1	2	3
$f(x)$	1	3	5	7

(iii) Arrow Diagram

Let us represent f by an arrow diagram.

We draw two closed curves to represent the sets A and B.

Here each element of A and its unique image element in B are related with an arrow.

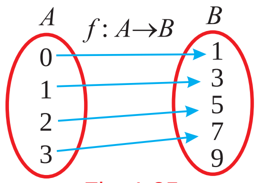


Fig. 1.25

(iv) Graph

We are given that

$$f = \{(x, f(x)) \mid x \in A\} = \{(0,1), (1,3), (2,5), (3,7)\}.$$

Now, the points (0, 1), (1, 3), (2, 5) and (3, 7) are plotted on the plane as shown below.

The totality of all points represent the graph of the function.

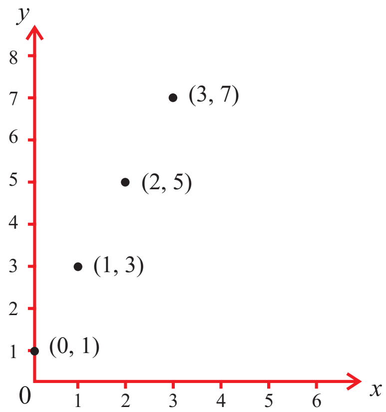


Fig. 1.26

1.8.3 Types of functions

Based on some properties of a function, we divide functions into certain types.

(i) One-One function

Let $f : A \to B$ be a function. The function f is called an **one-one** function if it takes different elements of A into different elements of B. That is, we say f is one-one if $u \neq v$ in A always imply $f(u) \neq f(v)$. In other words f is one-one if no element in B is associated with more than one element in A.

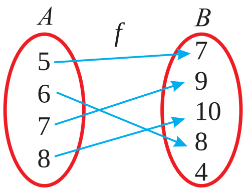


Fig. 1.27

A one-one function is also called an injective function. The above figure represents a one-one function.

(ii) Onto function

A function $f : A \to B$ is said to be an onto function if every element in B has a pre-image in A. That is, a function f is onto if for each $b \in B$, there is atleast one element $a \in A$, such that $f(a) = b$. This is same as saying that B is the range of f. An onto function is also called a surjective function. In the above figure, f is an onto function.

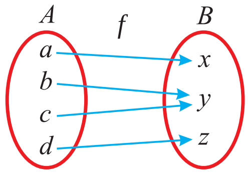


Fig. 1.28

(iii) One-One and onto function

A function $f : A \to B$ is called a one-one and onto or a bijective function if f is both a one-one and an onto function. Thus $f : A \to B$ is one-one and onto if f maps distinct elements of A into distinct images in B and every element in B is an image of some element in A.

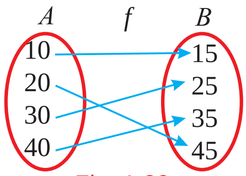


Fig. 1.29

Note

(i) A function $f : A \to B$ is onto if and only if B = range of f.

(ii) $f : A \to B$ is one-one and onto, if and only if $f(a_1) = f(a_2)$ implies $a_1 = a_2$ in A and every element in B has exactly one pre-image in A.

(iii) If $f : A \to B$ is a bijective function and if A and B are finite sets, then the cardinalities of A and B are same. In Fig.1.29, the function f is one - one and onto.

(iv) If $f : A \to B$ is a bijective function, then A and B are equivalent sets

(v) A one-one and onto function is also called a one-one correspondence.

(iv) Constant function

A function $f : A \to B$ is said to be a constant function if every element of A has the same image in B.

Range of a constant function is a singleton set.

Let $A = \{x, y, u, v, 1\}$, $B = \{3, 5, 7, 8, 10, 15\}$.

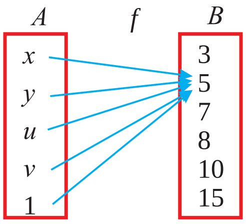


Fig. 1.30

The function $f : A \to B$ defined by $f(x) = 5$ for every $x \in A$ is a constant function.

The given figure represents a constant function.

(v) Identity function

Let A be a non-empty set. A function $f : A \to A$ is called an identity function of A if $f(a) = a$ for all $a \in A$. That is, an identity function maps each element of A into itself.

For example, let $A = \mathbb{R}$. The function $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $f(x) = x$ for all $x \in \mathbb{R}$ is the identity function on $\mathbb{R}$. Fig.1.31 represents the graph of the identity function on $\mathbb{R}$.

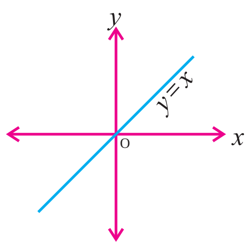


Fig. 1.31

Example 1.21

Let $A = \{1, 2, 3, 4, 5\}$, $B = \mathbb{N}$ and $f : A \to B$ be defined by $f(x) = x^2$. Find the range of f. Identify the type of function.

Solution Now, $A = \{1, 2, 3, 4, 5\}$; $B = \{1, 2, 3, 4, \cdots\}$

Given $f : A \to B$ and $f(x) = x^2$

$\therefore$ $f(1) = 1^2 = 1$; $f(2) = 4$; $f(3) = 9$; $f(4) = 16$; $f(5) = 25$.

Range of $f = \{1, 4, 9, 16, 25\}$

Since distinct elements are mapped into distinct images, it is a one-one function. However, the function is not onto, since $3 \in B$ but there is no $x \in A$ such that

$f(x) = x^2 = 3$.

Remarks

However, a function $g : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $g(x) = x^2$ is not one-one because, if $u = 1$ and $v = -1$ then $u \neq v$ but $g(u) = g(1) = 1 = g(-1) = g(v)$. So, just formula alone does not make a function one-one or onto. We need to consider the rule, its domain and co-domain in deciding one-to-one and onto.

Example 1.22

A function $f : [1, 6) \longrightarrow \mathbb{R}$ is defined as follows

$$f(x) = \begin{cases} 1 + x, & 1 \le x < 2 \\ 2x - 1, & 2 \le x < 4 \\ 3x^2 - 10, & 4 \le x < 6 \end{cases}$$

(Here, $[1, 6) = \{ x \in \mathbb{R} : 1 \le x < 6\}$)

Find the value of (i) $f(5)$ (ii) $f(3)$ (iii) $f(1)$

(iv) $f(2) - f(4)$ (v) $2f(5) - 3f(1)$

Solution

(i) Let us find $f(5)$. Since 5 lies between 4 and 6, we have to use $f(x) = 3x^2 - 10$.

Thus, $f(5) = 3(5^2) - 10 = 65$.

(ii) To find $f(3)$, note that 3 lies between 2 and 4.

So, we use $f(x) = 2x - 1$ to calculate $f(3)$.

Thus, $f(3) = 2(3) - 1 = 5$.

(iii) Let us find $f(1)$.

Now, 1 is in the interval $1 \le x < 2$

Thus, we have to use $f(x) = 1 + x$ to obtain $f(1) = 1 + 1 = 2$.

(iv) $f(2) - f(4)$

Now, 2 is in the interval $2 \le x < 4$ and so, we use $f(x) = 2x - 1$.

Thus, $f(2) = 2(2) - 1 = 3$.

Also, 4 is in the interval $4 \le x < 6$. Thus, we use $f(x) = 3x^2 - 10$.

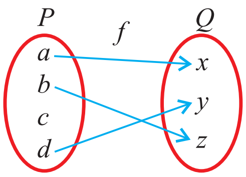
Therefore, $f(4) = 3(4^2) - 10 = 3(16) - 10 = 48 - 10 = 38$.

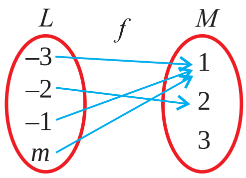
Hence, $f(2) - f(4) = 3 - 38 = -35$.

(v) To calculate $2\,f(5) - 3\,f(1)$, we shall make use of the values that we have already calculated in (i) and (iii). Thus, $2\,f(5) - 3\,f(1) = 2(65) - 3(2) = 130 - 6 = 124$.

Exercise 1.4

1. State whether each of the following arrow diagrams define a function or not. Justify your answer.

(i)


(ii)


2. For the given function $F = \{ (1, 3), (2, 5), (4, 7), (5, 9), (3, 1) \}$, write the domain and range.

3. Let $A = \{ 10, 11, 12, 13, 14 \}$; $B = \{ 0, 1, 2, 3, 5 \}$ and $f_i : A \to B$, $i = 1,2,3$. State the type of function for the following (give reason):

(i) $f_1 = \{ (10, 1), (11, 2), (12, 3), (13, 5), (14, 3) \}$

(ii) $f_2 = \{ (10, 1), (11, 1), (12, 1), (13, 1), (14, 1) \}$

(iii) $f_3 = \{ (10, 0), (11, 1), (12, 2), (13, 3), (14, 5) \}$

4. If $X = \{ 1, 2, 3, 4, 5 \}$, $Y = \{ 1, 3, 5, 7, 9 \}$ determine which of the following relations from X to Y are functions? Give reason for your answer. If it is a function, state its type.

(i) $R_1 = \{ (x, y) | y = x + 2,\ x \in X,\ y \in Y \}$

(ii) $R_2 = \{ (1, 1), (2, 1), (3, 3), (4, 3), (5, 5) \}$

(iii) $R_3 = \{ (1, 1), (1, 3), (3, 5), (3, 7), (5, 7) \}$

(iv) $R_4 = \{ (1, 3), (2, 5), (4, 7), (5, 9), (3, 1) \}$

5. If $R = \{(a, -2), (-5, b), (8, c), (d, -1)\}$ represents the identity function, find the values of a, b, c and d.

6. $A = \{-2, -1, 1, 2\}$ and $f = \left\{\left(x, \frac{1}{x}\right) : x \in A\right\}$. Write down the range of f. Is f a function from A to A ?

7. Let $f = \{(2, 7), (3, 4), (7, 9), (-1, 6), (0, 2), (5, 3)\}$ be a function from $A = \{-1, 0, 2, 3, 5, 7\}$ to $B = \{2, 3, 4, 6, 7, 9\}$. Is this (i) an one-one function (ii) an onto function (iii) both one-one and onto function?

8. Write the pre-images of 2 and 3 in the function $f = \{(12, 2), (13, 3), (15, 3), (14, 2), (17, 17)\}$.

9. The following table represents a function from $A = \{5, 6, 8, 10\}$ to $B = \{19, 15, 9, 11\}$ where $f(x) = 2x - 1$. Find the values of a and b.

x	5	6	8	10
$f(x)$	a	11	b	19

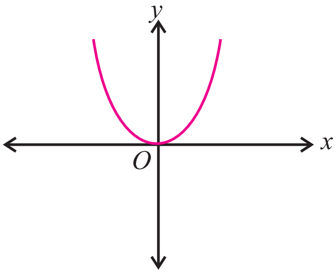
10. Let $A = \{5, 6, 7, 8\}$; $B = \{-11, 4, 7, -10, -7, -9, -13\}$ and $f = \{(x, y) : y = 3 - 2x,\ x \in A,\ y \in B\}$

(i) Write down the elements of f. (ii) What is the co-domain?

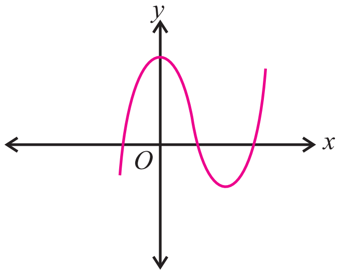
(iii) What is the range ? (iv) Identify the type of function.

11. State whether the following graphs represent a function. Give reason for your answer.

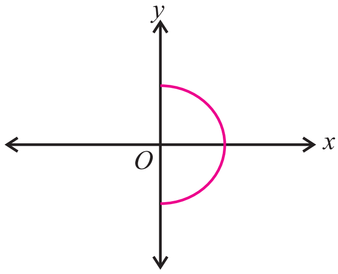
(i)



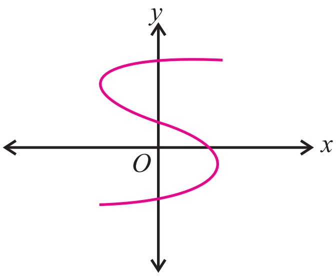
(ii)



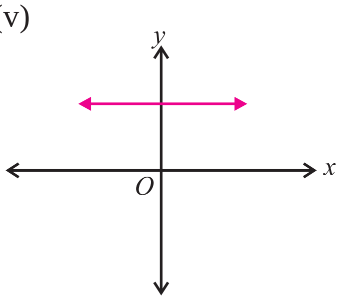
(iii)



(iv)



(v)



12. Represent the function $f = \{ (-1, 2), (-3, 1), (-5, 6), (-4, 3) \}$ as
(i) a table (ii) an arrow diagram

13. Let $A = \{ 6, 9, 15, 18, 21 \}$; $B = \{ 1, 2, 4, 5, 6 \}$ and $f : A \to B$ be defined by $f(x) = \frac{x-3}{3}$. Represent f by
(i) an arrow diagram (ii) a set of ordered pairs
(iii) a table (iv) a graph .

14. Let $A = \{4, 6, 8, 10 \}$ and $B = \{ 3, 4, 5, 6, 7 \}$. If $f : A \to B$ is defined by $f(x) = \frac{1}{2}x + 1$ then represent f by (i) an arrow diagram (ii) a set of ordered pairs and (iii) a table.

15. A function $f : [-3,7) \to \mathbb{R}$ is defined as follows

$$f(x) = \begin{cases} 4x^2 - 1; & -3 \leq x < 2 \\ 3x - 2; & 2 \leq x \leq 4. \\ 2x - 3; & 4 < x < 7 \end{cases}$$

Find (i) $f(5) + f(6)$ (ii) $f(1) - f(-3)$
(iii) $f(-2) - f(4)$ (iv) $\frac{f(3) + f(-1)}{2f(6) - f(1)}$.

16. A function $f : [-7,6) \to \mathbb{R}$ is defined as follows

$$f(x) = \begin{cases} x^2 + 2x + 1; & -7 \leq x < -5 \\ x + 5; & -5 \leq x \leq 2 \\ x - 1; & 2 < x < 6\ . \end{cases}$$

Find (i) $2\,f(-4) + 3\,f(2)$ (ii) $f(-7) - f(-3)$ (iii) $\frac{4\,f(-3) + 2\,f(4)}{f(-6) - 3\,f(1)}$.

Exercise 1.5

Choose the correct answer

1. For two sets A and B, $A \cup B = A$ if and only if
(A) $B \subseteq A$ (B) $A \subseteq B$ (C) $A \neq B$ (D) $A \cap B = \phi$

2. If $A \subset B$, then $A \cap B$ is
(A) B (B) $A \backslash B$ (C) A (D) $B \backslash A$

3. For any two sets P and Q, $P \cap Q$ is
(A) $\{x : x \in P \text{ or } x \in Q\}$ (B) $\{x : x \in P \text{ and } x \notin Q\}$
(C) $\{x : x \in P \text{ and } x \in Q\}$ (D) $\{x : x \notin P \text{ and } x \in Q\}$

4. If $A = \{ p, q, r, s \}$, $B = \{ r, s, t, u \}$, then $A \backslash B$ is

(A) $\{ p, q \}$ (B) $\{ t, u \}$ (C) $\{ r, s \}$ (D) $\{p, q, r, s \}$

5. If $n[p(A)] = 64$, then $n(A)$ is

(A) 6 (B) 8 (C) 4 (D) 5

6. For any three sets A, B and C, $A \cap (B \cup C)$ is

(A) $(A \cup B) \cup (B \cap C)$ (B) $(A \cap B) \cup (A \cap C)$

(C) $A \cup (B \cap C)$ (D) $(A \cup B) \cap (B \cup C)$

7. For any two sets A and B, $\{(A \setminus B) \cup (B \setminus A)\} \cap (A \cap B)$ is

(A) ϕ (B) $A \cup B$ (C) $A \cap B$ (D) $A' \cap B'$

8. Which one of the following is not true ?

(A) $A \setminus B = A \cap B'$ (B) $A \setminus B = A \cap B$

(C) $A \setminus B = (A \cup B) \cap B'$ (D) $A \setminus B = (A \cup B) \setminus B$

9. For any three sets A, B and C, $B \setminus (A \cup C)$ is

(A) $(A \setminus B) \cap (A \setminus C)$ (B) $(B \setminus A) \cap (B \setminus C)$

(C) $(B \setminus A) \cap (A \setminus C)$ (D) $(A \setminus B) \cap (B \setminus C)$

10. If $n(A) = 20$, $n(B) = 30$ and $n(A \cup B) = 40$, then $n(A \cap B)$ is equal to

(A) 50 (B) 10 (C) 40 (D) 70.

11. If $\{ (x, 2), (4, y) \}$ represents an identity function, then (x, y) is

(A) (2, 4) (B) (4, 2) (C) (2, 2) (D) (4, 4)

12. If $\{ (7, 11), (5, a) \}$ represents a constant function, then the value of 'a' is

(A) 7 (B) 11 (C) 5 (D) 9

13. Given $f(x) = (-1)^x$ is a function from $\mathbb{N}$ to $\mathbb{Z}$. Then the range of f is

(A) $\{ 1\}$ (B) $\mathbb{N}$ (C) $\{ 1, -1 \}$ (D) $\mathbb{Z}$

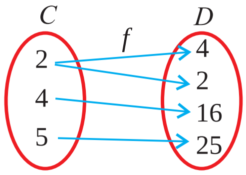
14. If $f = \{ (6, 3), (8, 9), (5, 3), (-1, 6) \}$, then the pre-images of 3 are

(A) 5 and -1 (B) 6 and 8 (C) 8 and -1 (D) 6 and 5.

15. Let $A = \{ 1, 3, 4, 7, 11 \}$, $B = \{-1, 1, 2, 5, 7, 9 \}$ and $f : A \to B$ be given by

$f = \{ (1, -1), (3, 2), (4, 1), (7, 5), (11, 9) \}$. Then f is

(A) one-one (B) onto (C) bijective (D) not a function

16. 

The given diagram represents

(A) an onto function (B) a constant function

(C) an one-one function (D) not a function

17. If $A = \{5, 6, 7\}$, $B = \{1, 2, 3, 4, 5\}$ and $f : A \to B$ is defined by $f(x) = x - 2$, then the range of f is

(A) { 1, 4, 5 } (B) { 1, 2, 3, 4, 5 } (C) { 2, 3, 4 } (D) { 3, 4, 5 }

18. If $f(x) = x^2 + 5$, then $f(-4) =$

(A) 26 (B) 21 (C) 20 (D) –20

19. If the range of a function is a singleton set, then it is

(A) a constant function (B) an identity function

(C) a bijective function (D) an one-one function

20. If $f : A \to B$ is a bijective function and if $n(A) = 5$, then $n(B)$ is equal to

(A) 10 (B) 4 (C) 5 (D) 25

Points to Remember

SETS

- A set is a collection of well defined objects.
  - Set union is commutative and associative.
  - Set intersection is commutative and associative.
  - Set difference is not commutative.
  - Set difference is associative only when the sets are mutually disjoint.
- Distributive Laws
  - $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
  - $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- De Morgan's Laws for set difference
  - $A \backslash (B \cup C) = (A \backslash B) \cap (A \backslash C)$
  - $A \backslash (B \cap C) = (A \backslash B) \cup (A \backslash C)$
- De Morgan's Laws for complementation.
  - $(A \cup B)' = A' \cap B'$
  - $(A \cap B)' = A' \cup B'$
- Formulae for the cardinality of union of sets
  - $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
  - $n(A \cup B \cup C)$
    $= n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$.

FUNCTIONS

- The cartesian product of A with B is defined as

  $A \times B = \{(a,b) \mid a \in A \text{ and } b \in B\}$.

- A relation R from A to B is a non-empty subset of $A \times B$. That is, $R \subseteq A \times B$.
- A function $f : X \to Y$ is defined if the following condition hold:

  Every $x \in X$ is associated with only one $y \in Y$.

- Every function can be represented by a graph. However, the converse is not true in general.
- If every vertical line intersects a graph in at most one point, then the graph represents a function.
- A function can be described by

  - a set of ordered pairs
  - an arrow diagram
  - a table and
  - a graph.

- The modulus or absolute value function $y = |x|$ is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

- Some types of functions:

  - One-One function (injective function) — (distinct elements have distinct images)
  - Onto function (surjective function) — (the range and the co-domain are equal)
  - Bijective function — (both one-one and onto)
  - Constant function — (range is a singleton set)
  - Identity function — (which leaves each input as it is)

Do you know?

The Millennium Prize problems are seven problems in Mathematics that were stated by the Clay Mathematics Institute in USA in 2000. As of August 2010, six of the problems remain unsolved. A correct solution to any of the problems results in a US \$1000,000 being awarded by the institute. Only Poincare conjecture has been solved by a Russian Mathematician Girigori Perelman in 2010. However, he declined the Millinnium Prize award. (Here, the word conjecture means a mathematical problem is to be proved or disproved)

2

SEQUENCES AND SERIES OF REAL NUMBERS

Mathematics is the Queen of Sciences, and arithmetic is the Queen of Mathematics - C.F.Gauss

- Introduction
- Sequences
- Arithmetic Progression (A.P.)
- Geometric Progression (G.P.)
- Series

Leonardo Pisano (Fibonacci)
(1170-1250)
Italy

Fibonacci played an important role in reviving ancient mathematics. His name is known to modern mathematicians mainly because of a number sequence named after him, known as the 'Fibonacci numbers', which he did not discover but used as an example.

2.1 Introduction

In this chapter, we shall learn about sequences and series of real numbers. Sequences are fundamental mathematical objects with a long history in mathematics. They are tools for the development of other concepts as well as tools for mathematization of real life situations.

Let us recall that the letters $\mathbb{N}$ and $\mathbb{R}$ denote the set of all positive integers and real numbers respectively.

Let us consider the following real-life situations.

(i) A team of ISRO scientists observes and records the height of a satellite from the sea level at regular intervals over a period of time.

(ii) The Railway Ministry wants to find out the number of people using Central railway station in Chennai on a daily basis and so it records the number of people entering the Central Railway station daily for 180 days.

(iii) A curious 9th standard student is interested in finding out all the digits that appear in the decimal part of the irrational number $\sqrt{5} = 2.236067978\cdots$ and writes down as

$2, 3, 6, 0, 6, 7, 9, 7, 8, \cdots$.

(iv) A student interested in finding all positive fractions with numerator 1, writes $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \cdots$.

(v) A mathematics teacher writes down the marks of her class according to alphabetical order of the students' names as $75, 95, 67, 35, 58, 47, 100, 89, 85, 60.$.

(vi) The same teacher writes down the same data in an ascending order as
35, 47, 58, 60, 67, 75, 85, 89, 95, 100.

In each of the above examples, some sets of real numbers have been listed in a specific order.

Note that in (iii) and (iv) the arrangements have infinite number of terms. In (i), (ii), (v) and (vi) there are only finite number of terms; but in (v) and (vi) the same set of numbers are written in different order.

2.2 Sequences

Definition

A sequence of real numbers is an arrangement or a list of real numbers in a specific order.

(i) If a sequence has only finite number of terms, then it is called a finite sequence.

(ii) If a sequence has infinitely many terms, then it is called an infinite sequence.

We denote a finite sequence as $S: a_1, a_2, a_3, \cdots, a_n$ or $S = \{a_j\}_{j=1}^{n}$ and an infinite sequence as $S: a_1, a_2, a_3, \cdots, a_n, \cdots$ or $S = \{a_j\}_{j=1}^{\infty}$ where a_k denotes the k^{th} term of the sequence. For example, a_1 denotes the first term and a_7 denotes the seventh term in the sequence.

Note that in the above examples, (i), (ii), (v) and (vi) are finite sequences, whereas (iii) and (iv) are infinite sequences

Observe that, when we say that a collection of numbers is listed in a sequence, we mean that the sequence has an identified first member, second member, third member and so on. We have already seen some examples of sequences. Let us consider some more examples below.

(i) 2, 4, 6, 8, $\cdots$, 2010. (finite number of terms)

(ii) 1, -1, 1, -1, 1, -1, 1, $\cdots$. (terms just keep oscillating between 1 and -1)

(iii) π, π, π, π, π. (terms are same; such sequences are constant sequences)

(iv) 2, 3, 5, 7, 11, 13, 17, 19, 23, $\cdots$. (list of all prime numbers)

(v) 0.3, 0.33, 0.333, 0.3333, 0.33333, $\cdots$. (infinite number of terms)

(vi) $S = \{a_n\}_1^{\infty}$ where $a_n = 1$ or 0 according to the outcome head or tail in the n^{th} toss of a coin.

From the above examples, (i) and (iii) are finite sequences and the other sequences are infinite sequences. One can easily see that some of them, i.e., (i) to (v) have a definite pattern or rule in the listing and hence we can find out any term in a particular position in the sequence. But in (vi), we cannot predict what a particular term is, however, we know it must be either 1 or 0. Here, we have used the word "pattern" to mean that the n^{th} term of a sequence is found based on the knowledge of its preceding elements in the sequence. In general, sequences can be viewed as functions.

2.2.1 Sequences viewed as functions

A finite real sequence $a_1, a_2, a_3, \cdots, a_n$ or $S = \{a_j\}_{j=1}^{n}$ can be viewed as a function $f : \{1, 2, 3, 4, \cdots, n\} \to \mathbb{R}$ defined by $f(k) = a_k,\; k = 1, 2, 3, \cdots, n.$

An infinite real sequence $a_1, a_2, a_3, \cdots, a_n, \cdots$ or $S = \{a_j\}_{j=1}^{\infty}$ can be viewed as a function $g : \mathbb{N} \to \mathbb{R}$ defined by $g(k) = a_k,\; \forall\, k \in \mathbb{N}$.

The symbol $\forall$ means "for all". If the general term a_k of a sequence $\{a_k\}_1^{\infty}$ is given, we can construct the whole sequence. Thus, a sequence is a function whose domain is the set{ 1, 2, 3, $\cdots$, }of natural numbers, or some subset of the natural numbers and whose range is a subset of real numbers.

Remarks

A function is not necessarily a sequence. For example, the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ given by $f(x) = 2x + 1, \forall\, x \in \mathbb{R}$ is not a sequence since the required listing is not possible. Also, note that the domain of f is not $\mathbb{N}$ or a subset $\{1, 2, \cdots, n\}$of $\mathbb{N}$.

Example 2.1

Write the first three terms in a sequence whose n^{th} term is given by

$$c_n = \frac{n(n+1)(2n+1)}{6}, \;\forall\, n \in \mathbb{N}$$

Solution Here, $$c_n = \frac{n(n+1)(2n+1)}{6}, \;\forall\, n \in \mathbb{N}$$

For $n = 1$, $$c_1 = \frac{1(1+1)(2(1)+1)}{6} = 1.$$

For $n = 2$, $$c_2 = \frac{2(2+1)(4+1)}{6} = \frac{2(3)(5)}{6} = 5.$$

Finally $n = 3$, $$c_3 = \frac{3(3+1)(7)}{6} = \frac{(3)(4)(7)}{6} = 14.$$

Hence, the first three terms of the sequence are 1, 5, and 14.

In the above example, we were given a formula for the general term and were able to find any particular term directly. In the following example, we shall see another way of generating a sequence.

Example 2.2

Write the first five terms of each of the following sequences.

(i) $a_1 = -1, \qquad a_n = \frac{a_{n-1}}{n+2}, \quad n > 1$ and $\forall\, n \in \mathbb{N}$

(ii) $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}, \quad n = 3, 4, \cdots.$

Solution

(i) Given $a_1 = -1$ and $a_n = \dfrac{a_{n-1}}{n+2},\quad n > 1$

$$a_2 = \frac{a_1}{2+2} = -\frac{1}{4}$$

$$a_3 = \frac{a_2}{3+2} = \frac{-\frac{1}{4}}{5} = -\frac{1}{20}$$

$$a_4 = \frac{a_3}{4+2} = \frac{-\frac{1}{20}}{6} = -\frac{1}{120}$$

$$a_5 = \frac{a_4}{5+2} = \frac{-\frac{1}{120}}{7} = -\frac{1}{840}$$

$\therefore$ The required terms of the sequence are $-1, -\frac{1}{4}, -\frac{1}{20}, -\frac{1}{120}$ and $-\frac{1}{840}$.

(ii) Given that $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$, for $n = 3, 4, 5, \cdots$.

Now, $F_1 = 1, F_2 = 1$

$$F_3 = F_2 + F_1 = 1 + 1 = 2$$

$$F_4 = F_3 + F_2 = 2 + 1 = 3$$

$$F_5 = F_4 + F_3 = 3 + 2 = 5$$

$\therefore$ The first five terms of the sequence are 1, 1, 2, 3, 5.

Remarks

The sequence given by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$, $n = 3, 4, \cdots$ is called the Fibonacci sequence. Its terms are listed as 1, 1, 2, 3, 5, 8, 13, 21, 34, $\cdots$. The Fibonacci sequence occurs in nature, like the arrangement of seeds in a sunflower. The number of spirals in the opposite directions of the seeds in a sunflower are consecutive numbers of the Fibonacci sequence.

Exercise 2.1

1. Write the first three terms of the following sequences whose n^{th} terms are given by

   (i) $a_n = \dfrac{n(n-2)}{3}$ (ii) $c_n = (-1)^n 3^{n+2}$ (iii) $z_n = \dfrac{(-1)^n n(n+2)}{4}$

2. Find the indicated terms in each of the sequences whose n^{th} terms are given by

   (i) $a_n = \dfrac{n+2}{2n+3}$; a_7, a_9 (ii) $a_n = (-1)^n 2^{n+3}(n+1)$; a_5, a_8

   (iii) $a_n = 2n^2 - 3n + 1$; a_5, a_7. (iv) $a_n = (-1)^n(1 - n + n^2)$; a_5, a_8

3. Find the 18th and 25th terms of the sequence defined by

$$a_n = \begin{cases} n(n+3), & \text{if } n \in \mathbb{N} \text{ and } n \text{ is even} \\ \dfrac{2n}{n^2+1}, & \text{if } n \in \mathbb{N} \text{ and } n \text{ is odd.} \end{cases}$$

4. Find the 13th and 16th terms of the sequence defined by

$$b_n = \begin{cases} n^2, & \text{if } n \in \mathbb{N} \text{ and } n \text{ is even} \\ n(n+2), & \text{if } n \in \mathbb{N} \text{ and } n \text{ is odd.} \end{cases}$$

5. Find the first five terms of the sequence given by

$$a_1 = 2, a_2 = 3 + a_1 \text{ and } a_n = 2a_{n-1} + 5 \quad \text{for } n > 2.$$

6. Find the first six terms of the sequence given by

$$a_1 = a_2 = a_3 = 1 \text{ and } a_n = a_{n-1} + a_{n-2} \quad \text{for } n > 3.$$

2.3 Arithmetic sequence or Arithmetic Progression (A.P.)

In this section we shall see some special types of sequences.

Definition

A sequence $a_1, a_2, a_3, \cdots, a_n, \cdots$ is called an arithmetic sequence if $a_{n+1} = a_n + d$, $n \in \mathbb{N}$ where d is a constant. Here a_1 is called the first term and the constant d is called the common difference. An arithmetic sequence is also called an Arithmetic Progression (A.P.).

Examples

(i) $2, 5, 8, 11, 14, \cdots$ is an A.P. because $a_1 = 2$ and the common difference $d = 3$.

(ii) $-4, -4, -4, -4, \cdots$ is an A.P. because $a_1 = -4$ and $d = 0$.

(iii) $2, 1.5, 1, 0.5, 0, -0.5, -1.0, -1.5, \cdots$ is an A.P. because $a_1 = 2$ and $d = -0.5$.

The general form of an A.P.

Let us understand the general form of an A.P. Suppose that a is the first term and d is the common difference of an arithmetic sequence $\{a_k\}_{k=1}^{\infty}$. Then, we have

$$a_1 = a \text{ and } a_{n+1} = a_n + d, \ \forall\, n \in \mathbb{N}.$$

For $n = 1, 2, 3$ we get,

$$a_2 = a_1 + d = a + d = a + (2-1)d$$
$$a_3 = a_2 + d = (a+d) + d = a + 2d = a + (3-1)d$$
$$a_4 = a_3 + d = (a+2d) + d = a + 3d = a + (4-1)d$$

Following the pattern, we see that the n^{th} term a_n as

$$a_n = a_{n-1} + d = [a + (n-2)d] + d = a + (n-1)d.$$

Thus, we have $a_n = a + (n-1)d$ for every $n \in \mathbb{N}$.

So, a typical arithmetic sequence or A.P. looks like

$$a,\ a+d,\ a+2d,\ a+3d,\ \cdots,\ a+(n-1)d,\ a+nd, \cdots$$

Also, the formula for the general term of an Arithmetic sequence is of the form

$$t_n = a + (n-1)d \quad \text{for every } n \in \mathbb{N}.$$

Note

(i) Remember a sequence may also be a finite sequence. So, if an A.P. has only n terms, then the last term l is given by $l = a + (n-1)d$

(ii) $l = a + (n-1)d$ can also be rewritten as $n = \left(\frac{l-a}{d}\right) + 1$. This helps us to find the number of terms when the first, the last term and the common difference are given.

(iii) Three consecutive terms of an A.P. may be taken as $m-d,\ m,\ m+d$

(iv) Four consecutive terms of an A.P. may be taken as $m-3d,\ m-d,\ m+d,\ m+3d$ with common difference $2d$.

(v) An A.P. remains an A.P. if each of its terms is added or subtracted by a same constant.

(vi) An A.P. remains an A.P. if each of its terms is multiplied or divided by a non-zero constant.

Example 2.3

Which of the following sequences are in an A.P.?

(i) $\frac{2}{3}, \frac{4}{5}, \frac{6}{7}, \cdots$. (ii) $3m-1, 3m-3, 3m-5, \cdots$.

Solution

(i) Let t_n, $n \in \mathbb{N}$ be the n^{th} term of the given sequence.

$\therefore$ $$t_1 = \frac{2}{3},\ t_2 = \frac{4}{5},\ t_3 = \frac{6}{7}$$

So $$t_2 - t_1 = \frac{4}{5} - \frac{2}{3} = \frac{12-10}{15} = \frac{2}{15}$$

$$t_3 - t_2 = \frac{6}{7} - \frac{4}{5} = \frac{30-28}{35} = \frac{2}{35}$$

Since $t_2 - t_1 \neq t_3 - t_2$, the given sequence is not an A.P.

(ii) Given $3m-1, 3m-3, 3m-5, \cdots$.

Here $t_1 = 3m-1,\ t_2 = 3m-3,\ t_3 = 3m-5, \cdots$.

$\therefore$ $t_2 - t_1 = (3m-3) - (3m-1) = -2$

Also, $t_3 - t_2 = (3m-5) - (3m-3) = -2$

Hence, the given sequence is an A.P. with first term $3m-1$ and the common difference -2.

Example 2.4

Find the first term and common difference of the A.P.

(i) $5, 2, -1, -4, \cdots$. (ii) $\frac{1}{2}, \frac{5}{6}, \frac{7}{6}, \frac{3}{2}, \cdots, \frac{17}{6}$

Solution

(i) First term $a = 5$, and the common difference $d = 2 - 5 = -3$.

(ii) $a = \frac{1}{2}$ and the common difference $d = \frac{5}{6} - \frac{1}{2} = \frac{5-3}{6} = \frac{1}{3}$.

Example 2.5

Find the smallest positive integer n such that t_n of the arithmetic sequence $20, 19\frac{1}{4}, 18\frac{1}{2}, \cdots$ is negative.?

Solution Here we have $a = 20$, $d = 19\frac{1}{4} - 20 = -\frac{3}{4}$.

We want to find the first positive integer n such that $t_n < 0$.

This is same as solving $a + (n-1)d < 0$ for smallest $n \in \mathbb{N}$.

That is solving $20 + (n-1)\left(-\frac{3}{4}\right) < 0$ for smallest $n \in \mathbb{N}$.

Now, $(n-1)\left(-\frac{3}{4}\right) < -20$

$\Longrightarrow (n-1) \times \frac{3}{4} > 20$ (The inequality is reversed on multiplying both sides by -1)

$\therefore \quad n - 1 > 20 \times \frac{4}{3} = \frac{80}{3} = 26\frac{2}{3}$.

This implies $n > 26\frac{2}{3} + 1$. That is, $n > 27\frac{2}{3} = 27.66$

Thus, the smallest positive integer $n \in \mathbb{N}$ satisfying the inequality is $n = 28$.

Hence, the 28th term, t_{28} is the first negative term of the A.P.

Example 2.6

In a flower garden, there are 23 rose plants in the first row, 21 in the second row, 19 in the third row and so on. There are 5 rose plants in the last row. How many rows are there in the flower garden?

Solution Let n be the number of rows in the flower garden .

The number of rose plants in the 1st, 2nd, 3rd $, \cdots, n^{th}$ rows are $23, 21, 19, \cdots, 5$ respectively.

Now, $t_k - t_{k-1} = -2$ for $k = 2, \cdots, n$.

Thus, the sequence $23, 21, 19, \cdots, 5$ is in an A.P.

We have $a = 23$, $d = -2$, and $l = 5$.

$$\therefore \quad n = \frac{l-a}{d} + 1 = \frac{5-23}{-2} + 1 = 10.$$

So, there are 10 rows in the flower garden.

Example 2.7

If a person joins his work in 2010 with an annual salary of ₹30,000 and receives an annual increment of ₹600 every year, in which year, will his annual salary be ₹39,000?

Solution Suppose that the person's annual salary reaches ₹39,000 in the n^{th} year.

Annual salary of the person in 2010, 2011, 2012, $\cdots$, $[2010 + (n-1)]$ will be

₹30,000, ₹30,600, ₹31,200, $\cdots$, ₹39000 respectively.

First note that the sequence of salaries form an A.P.

To find the required number of terms, let us divide each term of the sequence by a fixed constant 100. Now, we get the new sequence 300, 306, 312, $\cdots$, 390.

Here $a = 300$, $d = 6$, $l = 390$.

So, $n = \dfrac{l-a}{d} + 1$

$$= \frac{390-300}{6} + 1 = \frac{90}{6} + 1 = 16$$

Thus, 16th annual salary of the person will be ₹39,000.

$\therefore$ His annual salary will reach ₹39,000 in the year 2025.

Example 2.8

Three numbers are in the ratio $2 : 5 : 7$. If the first number, the resulting number on the substraction of 7 from the second number and the third number form an arithmetic sequence, then find the numbers.

Solution Let the numbers be $2x, 5x$ and $7x$ for some unknown $x, (x \neq 0)$

By the given information, we have that $2x, 5x-7, 7x$ are in A.P.

$\therefore \; (5x-7) - 2x = 7x - (5x-7) \implies 3x - 7 = 2x + 7$ and so $x = 14$.

Thus, the required numbers are 28, 70, 98.

Exercise 2.2

1. The first term of an A.P. is 6 and the common difference is 5. Find the A.P. and its general term.

2. Find the common difference and 15th term of the A.P. 125, 120, 115, 110, $\cdots$.

3. Which term of the arithmetic sequence $24, 23\frac{1}{4}, 22\frac{1}{2}, 21\frac{3}{4}, \cdots$ is 3?

4. Find the 12^{th} term of the A.P. $\sqrt{2}, 3\sqrt{2}, 5\sqrt{2}, \cdots$.

5. Find the 17^{th} term of the A.P. $4, 9, 14, \cdots$.

6. How many terms are there in the following Arithmetic Progressions?

   (i) $-1, -\frac{5}{6}, -\frac{2}{3}, \cdots, \frac{10}{3}$. (ii) $7, 13, 19, \cdots, 205$.

7. If 9^{th} term of an A.P. is zero, prove that its 29^{th} term is double (twice) the 19^{th} term.

8. The 10^{th} and 18^{th} terms of an A.P. are 41 and 73 respectively. Find the 27^{th} term.

9. Find n so that the n^{th} terms of the following two A.P.'s are the same.

   $1, 7, 13, 19, \cdots$ and $100, 95, 90, \cdots$.

10. How many two digit numbers are divisible by 13?

11. A TV manufacturer has produced 1000 TVs in the seventh year and 1450 TVs in the tenth year. Assuming that the production increases uniformly by a fixed number every year, find the number of TVs produced in the first year and in the 15^{th} year.

12. A man has saved ₹640 during the first month, ₹720 in the second month and ₹800 in the third month. If he continues his savings in this sequence, what will be his savings in the 25^{th} month?

13. The sum of three consecutive terms in an A.P. is 6 and their product is –120. Find the three numbers.

14. Find the three consecutive terms in an A. P. whose sum is 18 and the sum of their squares is 140.

15. If m times the m^{th} term of an A.P. is equal to n times its n^{th} term, then show that the $(m+n)^{th}$ term of the A.P. is zero.

16. A person has deposited ₹25,000 in an investment which yields 14% simple interest annually. Do these amounts (principal + interest) form an A.P.? If so, determine the amount of investment after 20 years.

17. If a, b, c are in A.P. then prove that $(a-c)^2 = 4(b^2 - ac)$.

18. If a, b, c are in A.P. then prove that $\frac{1}{bc}, \frac{1}{ca}, \frac{1}{ab}$ are also in A.P.

19. If a^2, b^2, c^2 are in A.P. then show that $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are also in A.P.

20. If $a^x = b^y = c^z, x \neq 0, y \neq 0, z \neq 0$ and $b^2 = ac$, then show that $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ are in A.P.

2.4 Geometric Sequence or Geometric Progression (G.P.)

Definition

A sequence $a_1, a_2, a_3, \cdots, a_n, \cdots$ is called a geometric sequence if $a_{n+1} = a_n r$, $n \in \mathbb{N}$, where r is a non-zero constant. Here, a_1 is the first term and the constant r is called the common ratio. A geometric sequence is also called a Geometric Progression (G.P.).

Let us consider some examples of geometric sequences.

(i) $3, 6, 12, 24, \cdots$.

A sequence $\{a_n\}_1^\infty$ is a geometric sequence if $\dfrac{a_{n+1}}{a_n} = r \neq 0, \; n \in \mathbb{N}$.

Now, $\dfrac{6}{3} = \dfrac{12}{6} = \dfrac{24}{12} = 2 \neq 0$. So the given sequence is a geometric sequence.

(ii) $\dfrac{1}{9}, -\dfrac{1}{27}, \dfrac{1}{81}, -\dfrac{1}{243}, \cdots$.

Here, we have $\dfrac{-\frac{1}{27}}{\frac{1}{9}} = \dfrac{\frac{1}{81}}{-\frac{1}{27}} = \dfrac{-\frac{1}{243}}{\frac{1}{81}} = -\dfrac{1}{3} \neq 0$.

Thus, the given sequence is a geometric sequence.

The general form of a G.P.

Let us derive the general form of a G.P. Suppose that a is the first term and r is the common ratio of a geometric sequence $\{a_k\}_{k=1}^\infty$. Then, we have

$$a_1 = a \quad \text{and} \quad \frac{a_{n+1}}{a_n} = r \quad \text{for } n \in \mathbb{N}.$$

Thus, $a_{n+1} = r\, a_n$ for $n \in \mathbb{N}$.

For $n = 1, 2, 3$ we get,

$$a_2 = a_1 r = ar = ar^{2-1}$$
$$a_3 = a_2 r = (ar)r = ar^2 = ar^{3-1}$$
$$a_4 = a_3 r = (ar^2)r = ar^3 = ar^{4-1}$$

Following the pattern, we have

$$a_n = a_{n-1} r = (ar^{n-2})r = ar^{n-1}.$$

Thus, $a_n = ar^{n-1}$ for every $n \in \mathbb{N}$, gives n^{th} term of the G.P.

So, a typical geometric sequence or G.P. looks like

$$a, ar, ar^2, ar^3, \cdots, ar^{n-1}, ar^n, \cdots.$$

Thus , the formula for the general term of a geometric sequence is

$$t_n = ar^{n-1}, \quad n = 1, 2, 3, \cdots.$$

Suppose we are given the first few terms of a sequence, how can we determine if the given sequence is a geometric sequence or not?

If $\frac{t_{n+1}}{t_n} = r, \forall\, n \in \mathbb{N}$, where r is a non-zero constant, then $\{t_n\}_1^{\infty}$ is in G.P.

Note

(i) If the ratio of any term other than the first term to its preceding term of a sequence is a non-zero constant, then it is a geometric sequence.

(ii) A geometric sequence remains a geometric sequence if each term is multiplied or divided by a non zero constant.

(iii) Three consecutive terms in a G.P may be taken as $\frac{a}{r}, a, ar$ with common ratio r.

(iv) Four consecutive terms in a G.P may be taken as $\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$. (here, the common ratio is r^2 not r as above)

Example 2.9

Which of the following sequences are geometric sequences

(i) $5, 10, 15, 20, \cdots$. (ii) $0.15, 0.015, 0.0015, \cdots$. (iii) $\sqrt{7}, \sqrt{21}, 3\sqrt{7}, 3\sqrt{21}, \cdots$.

Solution

(i) Considering the ratios of the consecutive terms, we see that $\frac{10}{5} \neq \frac{15}{10}$.

Thus, there is no common ratio. Hence it is not a geometric sequence.

(ii) We see that $\frac{0.015}{0.15} = \frac{0.0015}{0.015} = \cdots = \frac{1}{10}$.

Since the common ratio is $\frac{1}{10}$, the given sequence is a geometric sequence.

(iii) Now, $\frac{\sqrt{21}}{\sqrt{7}} = \frac{3\sqrt{7}}{\sqrt{21}} = \frac{3\sqrt{21}}{3\sqrt{7}} = \cdots = \sqrt{3}$. Thus, the common ratio is $\sqrt{3}$.

Therefore, the given sequence is a geometric sequence.

Example 2.10

Find the common ratio and the general term of the following geometric sequences.

(i) $\frac{2}{5}, \frac{6}{25}, \frac{18}{125}, \cdots$. (ii) $0.02, 0.006, 0.0018, \cdots$.

Solution

(i) Given sequence is a geometric sequence.

The common ratio is given by $r = \frac{t_2}{t_1} = \frac{t_3}{t_2} = \cdots$.

Thus, $r = \frac{\frac{6}{25}}{\frac{2}{5}} = \frac{3}{5}$.

The first term of the sequence is $\frac{2}{5}$. So, the general term of the sequence is

$$t_n = ar^{n-1}, \; n = 1, 2, 3, \cdots.$$

$$\Longrightarrow \qquad t_n = \frac{2}{5}\left(\frac{3}{5}\right)^{n-1}, \quad n = 1, 2, 3, \cdots$$

(ii) The common ratio of the given geometric sequence is

$$r = \frac{0.006}{0.02} = 0.3 = \frac{3}{10}.$$

The first term of the geometric sequence is 0.02

So, the sequence can be represented by

$$t_n = (0.02)\left(\frac{3}{10}\right)^{n-1}, \quad n = 1, 2, 3, \cdots$$

Example 2.11

The 4th term of a geometric sequence is $\frac{2}{3}$ and the seventh term is $\frac{16}{81}$. Find the geometric sequence.

Solution Given that $t_4 = \frac{2}{3}$ and $t_7 = \frac{16}{81}$.

Using the formula $t_n = ar^{n-1}, \; n = 1, 2, 3, \cdots$. for the general term we have,

$$t_4 = ar^3 = \frac{2}{3} \quad \text{and} \quad t_7 = ar^6 = \frac{16}{81}.$$

Note that in order to find the geometric sequence, we need to find a and r.

By dividing t_7 by t_4 we obtain,

$$\frac{t_7}{t_4} = \frac{ar^6}{ar^3} = \frac{\frac{16}{81}}{\frac{2}{3}} = \frac{8}{27}.$$

Thus, $\quad r^3 = \frac{8}{27} = \left(\frac{2}{3}\right)^3$ which implies $r = \frac{2}{3}$.

Now, $\quad t_4 = \frac{2}{3} \Longrightarrow ar^3 = \left(\frac{2}{3}\right)$.

$$\Longrightarrow \quad a\left(\frac{8}{27}\right) = \frac{2}{3}. \qquad \therefore \quad a = \frac{9}{4}.$$

Hence, the required geometric sequence is $a, \; ar, \; ar^2, \; ar^3, \; \cdots, \; ar^{n-1}, \; ar^n, \cdots$

That is, $\quad \frac{9}{4}, \; \frac{9}{4}\left(\frac{2}{3}\right), \; \frac{9}{4}\left(\frac{2}{3}\right)^2, \; \cdots.$

Example 2.12

The number of bacteria in a certain culture doubles every hour. If there were 30 bacteria present in the culture initially, how many bacteria will be present at the end of 14th hour?

Solution Note that the number of bacteria present in the culture doubles at the end of successive hours.

Number of bacteria present initially in the culture $= 30$

Number of bacteria present at the end of first hour $= 2(30)$

Number of bacteria present at the end of second hour $= 2(2(30)) = 30(2^2)$

Continuing in this way, we see that the number of bacteria present at the end of every hour forms a G.P. with the common ratio $r = 2$.

Thus, if t_n denotes the number of bacteria after n hours,

$$t_n = 30\,(2^n) \text{ is the general term of the G.P.}$$

Hence, the number of bacteria at the end of 14th hour is given by $t_{14} = 30\,(2^{14})$.

Example 2.13

An amount ₹500 is deposited in a bank which pays annual interest at the rate of 10% compounded annually. What will be the value of this deposit at the end of 10th year?

Solution

The principal is ₹500. So, the interest for this principal for one year is $500\left(\frac{10}{100}\right) = 50$.

Thus, the principal for the 2nd year = Principal for 1st year + Interest

$$= 500 + 500\left(\frac{10}{100}\right) = 500\left(1 + \frac{10}{100}\right)$$

Now, the interest for the second year $= \left(500\left(1 + \frac{10}{100}\right)\right)\left(\frac{10}{100}\right)$.

So, the principal for the third year $= 500\left(1 + \frac{10}{100}\right) + 500\left(1 + \frac{10}{100}\right)\frac{10}{100}$

$$= 500\left(1 + \frac{10}{100}\right)^2$$

Continuing in this way we see that the principal for the n^{th} year $\Big\} = 500\left(1 + \frac{10}{100}\right)^{n-1}$.

The amount at the end of $(n-1)^{\text{th}}$ year = Principal for the n^{th} year.

Thus, the amount in the account at the end of n^{th} year.

$$= 500\left(1 + \frac{10}{100}\right)^{n-1} + 500\left(1 + \frac{10}{100}\right)^{n-1}\left(\frac{10}{100}\right) = 500\left(\frac{11}{10}\right)^n.$$

The amount in the account at the end of 10th year

$$= ₹\ 500\left(1 + \frac{10}{100}\right)^{10} = ₹\ 500\left(\frac{11}{10}\right)^{10}.$$

Remarks

By using the above method, one can derive a formula for finding the total amount for compound interest problems. Derive the formula:

$$A = P(1 + i)^n$$

where A is the amount, P is the principal, $i = \frac{r}{100}$, r is the annual interest rate and n is the number of years.

Example 2.14

The sum of first three terms of a geometric sequence is $\frac{13}{12}$ and their product is -1. Find the common ratio and the terms.

Solution We may take the first three terms of the geometric sequence as $\frac{a}{r}, a, ar$.

Then, $$\frac{a}{r} + a + ar = \frac{13}{12}$$

$$a\left(\frac{1}{r} + 1 + r\right) = \frac{13}{12} \implies a\left(\frac{r^2 + r + 1}{r}\right) = \frac{13}{12} \qquad (1)$$

Also,

$$\left(\frac{a}{r}\right)(a)(ar) = -1$$

$$\implies \qquad a^3 = -1 \qquad \therefore \quad a = -1$$

Substituting $a = -1$ in (1) we obtain,

$$(-1)\left(\frac{r^2 + r + 1}{r}\right) = \frac{13}{12}$$

$$\implies \qquad 12r^2 + 12r + 12 = -13r$$

$$12r^2 + 25r + 12 = 0$$

$$(3r + 4)(4r + 3) = 0$$

Thus, $r = -\frac{4}{3}$ or $-\frac{3}{4}$

When $r = -\frac{4}{3}$ and $a = -1$, the terms are $\frac{3}{4}, -1, \frac{4}{3}$.

When $r = -\frac{3}{4}$ and $a = -1$, we get $\frac{4}{3}, -1, \frac{3}{4}$, which is in the reverse order.

Example 2.15

If a, b, c, d are in geometric sequence, then prove that

$(b - c)^2 + (c - a)^2 + (d - b)^2 = (a - d)^2$

Solution Given a, b, c, d are in a geometric sequence.

Let r be the common ratio of the given sequence. Here, the first term is a.

Thus, $b = ar, \quad c = ar^2, \quad d = ar^3$

Now, $(b - c)^2 + (c - a)^2 + (d - b)^2$

$$= (ar - ar^2)^2 + (ar^2 - a)^2 + (ar^3 - ar)^2$$

$$= a^2[(r - r^2)^2 + (r^2 - 1)^2 + (r^3 - r)^2]$$

$$= a^2[r^2 - 2r^3 + r^4 + r^4 - 2r^2 + 1 + r^6 - 2r^4 + r^2]$$

$$= a^2[r^6 - 2r^3 + 1] = a^2[r^3 - 1]^2$$

$$= (ar^3 - a)^2 = (a - ar^3)^2 = (a - d)^2$$

Exercise 2.3

1. Find out which of the following sequences are geometric sequences. For those geometric sequences, find the common ratio.

   (i) $0.12,\ 0.24,\ 0.48, \cdots .$ (ii) $0.004,\ 0.02,\ 0.1, \cdots .$ (iii) $\frac{1}{2}, \frac{1}{3}, \frac{2}{9}, \frac{4}{27}, \cdots .$

   (iv) $12,\ 1,\ \frac{1}{12}, \cdots .$ (v) $\sqrt{2}, \frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}, \cdots .$ (vi) $4, -2, -1, -\frac{1}{2}, \cdots .$

2. Find the 10^{th} term and common ratio of the geometric sequence $\frac{1}{4}, -\frac{1}{2}, 1, -2, \cdots .$

3. If the 4^{th} and 7^{th} terms of a G.P. are 54 and 1458 respectively, find the G.P.

4. In a geometric sequence, the first term is $\frac{1}{3}$ and the sixth term is $\frac{1}{729}$, find the G.P.

5. Which term of the geometric sequence,

   (i) $5,\ 2,\ \frac{4}{5}, \frac{8}{25}, \cdots$, is $\frac{128}{15625}$? (ii) $1,\ 2,\ 4,\ 8, \cdots$, is 1024 ?

6. If the geometric sequences $162, 54, 18, \cdots .$ and $\frac{2}{81}, \frac{2}{27}, \frac{2}{9}, \cdots$ have their n^{th} term equal, find the value of n.

7. The fifth term of a G.P. is 1875. If the first term is 3, find the common ratio.

8. The sum of three terms of a geometric sequence is $\frac{39}{10}$ and their product is 1. Find the common ratio and the terms.

9. If the product of three consecutive terms in G.P. is 216 and sum of their products in pairs is 156, find them.

10. Find the first three consecutive terms in G.P. whose sum is 7 and the sum of their reciprocals is $\frac{7}{4}$

11. The sum of the first three terms of a G.P. is 13 and sum of their squares is 91. Determine the G.P.

12. If ₹1000 is deposited in a bank which pays annual interest at the rate of 5% compounded annually, find the maturity amount at the end of 12 years .

13. A company purchases an office copier machine for ₹50,000. It is estimated that the copier depreciates in its value at a rate of 15% per year. What will be the value of the copier after 15 years?

14. If a, b, c, d are in a geometric sequence, then show that
$$(a - b + c)(b + c + d) = ab + bc + cd.$$

15. If a, b, c, d are in a G.P., then prove that $a + b,\ \ b + c,\ \ c + d,$ are also in G.P.

2.5 Series

Let us consider the following problem:

A person joined a job on January 1, 1990 at an annual salary of ₹25,000 and received an annual increment of ₹500 each year. What is the total salary he has received upto January 1, 2010?

First of all note that his annual salary forms an arithmetic sequence

$$25000,\ 25500,\ 26000,\ 26500, \cdots,\ (25000+19(500)).$$

To answer the above question, we need to add all of his twenty years salary. That is,

$$25000+25500+26000+26500+\cdots+(25000+19(500)).$$

So, we need to develop an idea of summing terms of a sequence.

Definition

An expression of addition of terms of a sequence is called a series.
If a series consists only a finite number of terms, it is called a finite series.
If a series consists of infinite number of terms of a sequence, it is called an infinite series.

Consider a sequence $S=\{a_n\}_{n=1}^{\infty}$ of real numbers. For each $n \in \mathbb{N}$ we define the partial sums by $S_n = a_1+a_2+,\cdots+a_n$, $n=1,2,3,\cdots$. Then $\{S_n\}_{n=1}^{\infty}$ is the sequence of partial sums of the given sequence $\{a_n\}_{n=1}^{\infty}$.

The ordered pair $(\{a_n\}_{n=1}^{\infty}, \{S_n\}_{n=1}^{\infty})$ is called an infinite series of terms of the sequence $\{a_n\}_1^{\infty}$. The infinite series is denoted by $a_1+a_2+a_3+\cdots$, or simply $\sum_{n=1}^{\infty} a_n$ where the symbol $\sum$ stands for summation and is pronounced as sigma.

Well, we can easily understand finite series (adding finite number of terms). It is impossible to add all the terms of an infinite sequence by the ordinary addition, since one could never complete the task. How can we understand (or assign a meaning to) adding infinitely many terms of a sequence? We will learn about this in higher classes in mathematics. For now we shall focus mostly on finite series.

In this section , we shall study Arithmetic series and Geometric series.

2.5.1 Arithmetic series

An arithmetic series is a series whose terms form an arithmetic sequence.

Sum of first n terms of an arithmetic sequence

Consider an arithmetic sequence with first term a and common difference d given by $a,\ a+d,\ a+2d, ...,\ a+(n-1)d, \cdots$.

Let S_n be the sum of first n terms of the arithmetic sequence.

Thus, $S_n = a + (a+d) + (a+2d) + \cdots (a + (n-1)d)$

$\Longrightarrow$ $S_n = na + (d + 2d + 3d + \cdots + (n-1)d)$

$= na + d(1 + 2 + 3 + \cdots + (n-1))$

So, we can simplify this formula if we can find the sum $1 + 2 + \cdots + (n-1)$.

This is nothing but the sum of the arithmetic sequence $1, 2, 3, \cdots, (n-1)$.

So, first we find the sum $1 + 2 + \cdots + (n-1)$ below.

Now, let us find the sum of the first n positive integers.

Let $S_n = 1 + 2 + 3 + \cdots + (n-2) + (n-1) + n$. (1)

We shall use a small trick to find the above sum. Note that we can write S_n also as

$$S_n = n + (n-1) + (n-2) + \cdots + 3 + 2 + 1. \quad (2)$$

Adding (1) and (2) we obtain,

$$2S_n = (n+1) + (n+1) + \cdots + (n+1) + (n+1). \quad (3)$$

Now, how many $(n+1)$ are there on the right hand side of (3)?

There are n terms in each of (1) and (2). We merely added corresponding terms from (1) and (2).

Thus, there must be exactly n such $(n+1)$'s.

Therefore, (3) simplifies to $2S_n = n(n+1)$.

Hence, the sum of the first n positive integers is given by

$$S_n = \frac{n(n+1)}{2}. \qquad \text{So, } 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}. \quad (4)$$

This is a useful formula in finding the sums.

Remarks

The above method was first used by the famous German mathematician **Carl Fredrick Gauss,** known as **Prince of Mathematics,** to find the sum of positive integers upto 100. This problem was given to him by his school teacher when he was just five years old. When you go to higher studies in mathematics, you will learn other methods to arrive at the above formula.

Carl Fredrick Gauss
(1777 – 1855)

Now, let us go back to summing first n terms of a general arithmetic sequence. We have already seen that

$$S_n = na + [d + 2d + 3d + \cdots + (n-1)d]$$

$$= na + d[1 + 2 + 3 + \cdots + (n-1)]$$

$$= na + d\frac{n(n-1)}{2} \quad \text{using (4)}$$

$$= \frac{n}{2}[2a + (n-1)d] \quad (5)$$

Hence, we have

$$S_n = \frac{n}{2}[a + (a + (n-1)d)] = \frac{n}{2}\text{ (first term + last term)}$$
$$= \frac{n}{2}(a + l).$$

The sum S_n of the first n terms of an arithmetic sequence with first term a is given by

(i) $S_n = \frac{n}{2}[2a + (n-1)d]$ if the common difference d is given.

(ii) $S_n = \frac{n}{2}(a + l)$, if the last term l is given.

Example 2.16

Find the sum of the arithmetic series $5 + 11 + 17 + \cdots + 95$.

Solution Given that the series $5 + 11 + 17 + \cdots + 95$ is an arithmetic series.

Note that $a = 5$, $d = 11 - 5 = 6$, $l = 95$.

Now, $$n = \frac{l-a}{d} + 1$$
$$= \frac{95-5}{6} + 1 = \frac{90}{6} + 1 = 16.$$

Hence, the sum $$S_n = \frac{n}{2}[l + a]$$
$$S_{16} = \frac{16}{2}[95 + 5] = 8(100) = 800.$$

Example 2.17

Find the sum of the first $2n$ terms of the following series.

$$1^2 - 2^2 + 3^2 - 4^2 + \ldots.$$

Solution We want to find $1^2 - 2^2 + 3^2 - 4^2 + \cdots$ to $2n$ terms

$= 1 - 4 + 9 - 16 + 25 - \cdots$ to $2n$ terms

$= (1-4) + (9-16) + (25-36) + \cdots$ to n terms. (after grouping)

$= -3 + (-7) + (-11) + \cdots$ n terms

Now, the above series is in an A.P. with first term $a = -3$ and common difference $d = -4$

Therefore, the required sum $= \frac{n}{2}[2a + (n-1)d]$

$$= \frac{n}{2}[2(-3) + (n-1)(-4)]$$
$$= \frac{n}{2}[-6 - 4n + 4] = \frac{n}{2}[-4n - 2]$$
$$= \frac{-2n}{2}(2n + 1) = -n(2n + 1).$$

Example 2.18

In an arithmetic series, the sum of first 14 terms is -203 and the sum of the next 11 terms is –572. Find the arithmetic series.

Solution Given that

$$
\begin{aligned}
S_{14} &= -203 \\
\Longrightarrow \quad \frac{14}{2}[2a + 13d] &= -203 \\
\Longrightarrow \quad 7[2a + 13d] &= -203 \\
\Longrightarrow \quad 2a + 13d &= -29. \qquad (1)
\end{aligned}
$$

Also, the sum of the next 11 terms $= -572$.

$$
\begin{aligned}
\text{Now,} \quad S_{25} &= S_{14} + (-572) \\
\text{That is,} \quad S_{25} &= -203 - 572 = -775. \\
\Longrightarrow \quad \frac{25}{2}[2a + 24d] &= -775 \\
\Longrightarrow \quad 2a + 24d &= -31 \times 2 \\
\Longrightarrow \quad a + 12d &= -31 \qquad (2)
\end{aligned}
$$

Solving (1) and (2) we get, $a = 5$ and $d = -3$.

Thus, the required arithmetic series is $5 + (5-3) + (5 + 2(-3)) + \cdots$.

That is, the series is $5 + 2 - 1 - 4 - 7 - \cdots$.

Example 2.19

How many terms of the arithmetic series $24 + 21 + 18 + 15 + \cdots$, be taken continuously so that their sum is -351.

Solution In the given arithmetic series, $a = 24$, $d = -3$.

$$
\begin{aligned}
\text{Let us find } n \text{ such that} \quad S_n &= -351 \\
\text{Now,} \quad S_n &= \frac{n}{2}[2a + (n-1)d] = -351 \\
\text{That is,} \quad \frac{n}{2}[2(24) + (n-1)(-3)] &= -351 \\
\Longrightarrow \quad \frac{n}{2}[48 - 3n + 3] &= -351 \\
\Longrightarrow \quad n(51 - 3n) &= -702 \\
\Longrightarrow \quad n^2 - 17n - 234 &= 0 \\
(n-26)(n+9) &= 0 \\
\therefore \quad n = 26 &\text{ or } n = -9
\end{aligned}
$$

Here n, being the number of terms needed, cannot be negative.

Thus, 26 terms are needed to get the sum -351.

Example 2.20

Find the sum of all 3 digit natural numbers, which are divisible by 8.

Solution

The three digit natural numbers divisible by 8 are 104, 112, 120, $\cdots$, 992.

Let S_n denote their sum. That is, $S_n = 104 + 112 + 120 + 128 + , \cdots + 992$.

Now, the sequence 104, 112, 120, $\cdots$, 992 forms an A.P.

Here, $a = 104,\ d = 8$ and $l = 992$.

$$\therefore \quad n = \frac{l-a}{d} + 1 = \frac{992-104}{8} + 1$$

$$= \frac{888}{8} + 1 = 112.$$

Thus, $S_{112} = \frac{n}{2}[a + l] = \frac{112}{2}[104 + 992] = 56(1096) = 61376$.

Hence, the sum of all three digit numbers, which are divisible by 8 is equal to 61376.

Example 2.21

The measures of the interior angles taken in order of a polygon form an arithmetic sequence. The least measurement in the sequence is 85°. The greatest measurement is 215°. Find the number of sides in the given polygon.

Solution Let n denote the number of sides of the polygon.

Now, the measures of interior angles form an arithmetic sequence.

Let the sum of the interior angles of the polygon be

$$S_n = a + (a+d) + (a+2d) + \cdots + l \text{, where } a = 85 \text{ and } l = 215.$$

We have, $S_n = \frac{n}{2}[l + a]$ (1)

We know that the sum of the interior angles of a polygon is $(n-2)\times 180^0$.

Thus, $S_n = (n-2)\times 180$

From (1), we have $\frac{n}{2}[l + a] = (n-2)\times 180$

$$\Longrightarrow \quad \frac{n}{2}[215 + 85] = (n-2)\times 180$$

$$150n = 180(n-2) \quad \Longrightarrow \quad n = 12..$$

Hence, the number of sides of the polygon is 12.

Exercise 2.4

1. Find the sum of the first (i) 75 positive integers (ii) 125 natural numbers.
2. Find the sum of the first 30 terms of an A.P. whose n^{th} term is $3 + 2n$.
3. Find the sum of each arithmetic series

   (i) $38 + 35 + 32 + \cdots + 2$. (ii) $6 + 5\frac{1}{4} + 4\frac{1}{2} + \cdots$ 25 terms.

4. Find the S_n for the following arithmetic series described.

   (i) $a = 5$, $n = 30$, $l = 121$ (ii) $a = 50$, $n = 25$, $d = -4$

5. Find the sum of the first 40 terms of the series $1^2 - 2^2 + 3^2 - 4^2 + \cdots$.

6. In an arithmetic series, the sum of first 11 terms is 44 and that of the next 11 terms is 55. Find the arithmetic series.

7. In the arithmetic sequence 60, 56, 52, 48,$\cdots$, starting from the first term, how many terms are needed so that their sum is 368?

8. Find the sum of all 3 digit natural numbers, which are divisible by 9.

9. Find the sum of first 20 terms of the arithmetic series in which 3rd term is 7 and 7th term is 2 more than three times its 3rd term.

10. Find the sum of all natural numbers between 300 and 500 which are divisible by 11.

11. Solve: $1 + 6 + 11 + 16 + \cdots + x = 148$.

12. Find the sum of all numbers between 100 and 200 which are not divisible by 5.

13. A construction company will be penalised each day for delay in construction of a bridge. The penalty will be ₹4000 for the first day and will increase by ₹1000 for each following day. Based on its budget, the company can afford to pay a maximum of ₹1,65,000 towards penalty. Find the maximum number of days by which the completion of work can be delayed

14. A sum of ₹1000 is deposited every year at 8% simple interest. Calculate the interest at the end of each year. Do these interest amounts form an A.P.? If so, find the total interest at the end of 30 years.

15. The sum of first n terms of a certain series is given as $3n^2 - 2n$. Show that the series is an arithmetic series.

16. If a clock strikes once at 1 o'clock, twice at 2 o'clock and so on, how many times will it strike in a day?

17. Show that the sum of an arithmetic series whose first term is a, second term b and the last term is c is equal to $\dfrac{(a+c)(b+c-2a)}{2(b-a)}$.

18. If there are $(2n+1)$ terms in an arithmetic series, then prove that the ratio of the sum of odd terms to the sum of even terms is $(n+1):n$.

19. The ratio of the sums of first m and first n terms of an arithmetic series is $m^2 : n^2$ show that the ratio of the m^{th} and n^{th} terms is $(2m-1):(2n-1)$

20. A gardener plans to construct a trapezoidal shaped structure in his garden. The longer side of trapezoid needs to start with a row of 97 bricks. Each row must be decreased by 2 bricks on each end and the construction should stop at 25^{th} row. How many bricks does he need to buy?

2.5.2 Geometric series

A series is a geometric series if the terms of the series form a geometric sequence.

Let $a, ar, ar^2, \cdots, ar^{n-1}, ar^n, \cdots$ be a geometric sequence where $r \neq 0$ is the common ratio. We want to find the sum of the first n terms of this sequence.

Let $S_n = a + ar + ar^2 + \cdots + ar^{n-1}$ (1)

If $r = 1$, then from (1) it follows that $S_n = na$.

For $r \neq 1$, using (1) we have

$$rS_n = r(a + ar + ar^2 + \cdots + ar^{n-1}) = ar + ar^2 + ar^3 + \cdots + ar^n. \quad (2)$$

Now subtracting (2) from (1), we get

$$S_n - rS_n = (a + ar + ar^2 + \cdots + ar^{n-1}) - (ar + ar^2 + \cdots + ar^n)$$

$$\Longrightarrow \quad S_n(1 - r) = a(1 - r^n)$$

Hence, we have $S_n = \dfrac{a(1 - r^n)}{1 - r}$, since $r \neq 1$.

The sum of the first n terms of a geometric series is given by

$$S_n = \begin{cases} \dfrac{a(r^n - 1)}{r - 1} = \dfrac{a(1 - r^n)}{1 - r}, & \text{if } r \neq 1 \\ na & \text{if } r = 1. \end{cases}$$

where a is the first term and r is the common ratio.

Remarks

Actually, if $-1 < r < 1$, then the following formula holds:

$$a + ar + ar^2 + \cdots + ar^n + \cdots = \frac{a}{1 - r}.$$

Note that the sum of infinite number of positive numbers may give a finite value.

Example 2.22

Find the sum of the first 25 terms of the geometric series

$16 - 48 + 144 - 432 + \cdots$.

Solution Here, $a = 16$, $r = -\dfrac{48}{16} = -3 \neq 1$. Now, $S_n = \dfrac{a(1 - r^n)}{1 - r}$, $r \neq 1$.

So, we have $S_{25} = \dfrac{16(1 - (-3)^{25})}{1 - (-3)} = \dfrac{16(1 + 3^{25})}{4} = 4(1 + 3^{25})$.

Example 2.23

Find S_n for each of the geometric series described below:

(i) $a = 2,\ t_6 = 486,\ n = 6$ (ii) $a = 2400,\ r = -3,\ n = 5$

Solution

(i) Here $a = 2,\ t_6 = 486,\ n = 6$

Now $t_6 = 2(r)^5 = 486$

$$\implies r^5 = 243 \quad \therefore \quad r = 3.$$

$$\text{Now,} \quad S_n = \frac{a(r^n - 1)}{r-1} \text{ if } r \neq 1$$

$$\text{Thus,} \quad S_6 = \frac{2(3^6 - 1)}{3-1} = 3^6 - 1 = 728.$$

(ii) Here $a = 2400,\ r = -3,\ n = 5$

$$\text{Thus,} \quad S_5 = \frac{a(r^5 - 1)}{r-1} \text{ if } r \neq 1$$

$$= \frac{2400[(-3)^5 - 1]}{(-3) - 1}$$

$$\text{Hence,} \quad S_5 = \frac{2400}{4}(1 + 3^5) = 600(1 + 243) = 146400.$$

Example 2.24

In the geometric series $2 + 4 + 8 + \cdots$, starting from the first term how many consecutive terms are needed to yield the sum 1022?

Solution Given the geometric series is $2 + 4 + 8 + \cdots$.

Let n be the number of terms required to get the sum.

Here $a = 2,\ r = 2,\ S_n = 1022$.

To find n, let us consider

$$S_n = \frac{a[r^n - 1]}{r-1} \text{ if } r \neq 1$$

$$= (2)\left[\frac{2^n - 1}{2-1}\right] = 2(2^n - 1).$$

But $S_n = 1022$ and hence $2(2^n - 1) = 1022$

$$\implies 2^n - 1 = 511$$

$$\implies 2^n = 512 = 2^9. \quad \text{Thus,} \quad n = 9.$$

Example 2.25

The first term of a geometric series is 375 and the fourth term is 192. Find the common ratio and the sum of the first 14 terms.

Solution Let a be the first term and r be the common ratio of the given G.P.

Given that $a = 375$, $t_4 = 192$.

Now, $t_n = ar^{n-1}$

$\therefore$ $t_4 = 375\,r^3 \implies 375\,r^3 = 192$

$r^3 = \frac{192}{375} \implies r^3 = \frac{64}{125}$

$r^3 = \left(\frac{4}{5}\right)^3 \implies r = \frac{4}{5}$, which is the required common ratio.

Now, $S_n = a\left[\frac{r^n - 1}{r - 1}\right]$ if $r \neq 1$

Thus, $S_{14} = \dfrac{375\left[\left(\frac{4}{5}\right)^{14} - 1\right]}{\frac{4}{5} - 1} = (-1)\times 5\times 375\left[\left(\frac{4}{5}\right)^{14} - 1\right]$

$= (375)(5)\left[1 - \left(\frac{4}{5}\right)^{14}\right] = 1875\left[1 - \left(\frac{4}{5}\right)^{14}\right]$.

Note

In the above example, one can use $S_n = a\left[\frac{1 - r^n}{1 - r}\right]$ if $r \neq 1$ instead of $S_n = a\left[\frac{r^n - 1}{r - 1}\right]$ if $r \neq 1$.

Example 2.26

A geometric series consists of four terms and has a positive common ratio. The sum of the first two terms is 8 and the sum of the last two terms is 72. Find the series.

Solution Let the sum of the four terms of the geometric series be $a + ar + ar^2 + ar^3$ and $r > 0$

Given that $a + ar = 8$ and $ar^2 + ar^3 = 72$

Now, $ar^2 + ar^3 = r^2(a + ar) = 72$

$\implies r^2(8) = 72 \quad \therefore \quad r = \pm 3$

Since $r > 0$, we have $r = 3$.

Now, $a + ar = 8 \implies a = 2$

Thus, the geometric series is $2 + 6 + 18 + 54$.

Example 2.27

Find the sum to n terms of the series $6 + 66 + 666 + \cdots$

Solution Note that the given series is not a geometric series.

We need to find $S_n = 6 + 66 + 666 + \cdots$ to n terms

$S_n = 6(1 + 11 + 111 + \cdots$ to n terms)

$= \frac{6}{9}(9 + 99 + 999 + \cdots$ to n terms) (Multiply and divide by 9)

$= \frac{2}{3}[(10 - 1) + (100 - 1) + (1000 - 1) + \cdots$ to n terms]

$= \frac{2}{3}[(10 + 10^2 + 10^3 + \cdots n$ terms$) - n]$

Thus, $S_n = \frac{2}{3}\left[\frac{10(10^n - 1)}{9} - n\right]$.

Example 2.28

An organisation plans to plant saplings in 25 streets in a town in such a way that one sapling for the first street, two for the second, four for the third, eight for the fourth street and so on. How many saplings are needed to complete the work?

Solution The number of saplings to be planted for each of the 25 streets in the town forms a G.P. Let S_n be the total number of saplings needed.

Then, $S_n = 1 + 2 + 4 + 8 + 16 + \cdots$ to 25 terms.

Here, $a = 1, \quad r = 2, \quad n = 25$

$$S_n = a\left[\frac{r^n - 1}{r - 1}\right]$$

$$S_{25} = (1)\frac{[2^{25} - 1]}{2 - 1}$$

$$= 2^{25} - 1$$

Thus, the number of saplings to be needed is $2^{25} - 1$.

Exercise 2.5

1. Find the sum of the first 20 terms of the geometric series $\frac{5}{2} + \frac{5}{6} + \frac{5}{18} + \cdots$.
2. Find the sum of the first 27 terms of the geometric series $\frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots$.
3. Find S_n for each of the geometric series described below.

   (i) $a = 3, \ t_8 = 384, \ n = 8$. (ii) $a = 5, \ r = 3, \ n = 12$.
4. Find the sum of the following finite series

   (i) $1 + 0.1 + 0.01 + 0.001 + \cdots + (0.1)^9$ (ii) $1 + 11 + 111 + \cdots$ to 20 terms.
5. How many consecutive terms starting from the first term of the series

   (i) $3 + 9 + 27 + \cdots$ would sum to 1092? (ii) $2 + 6 + 18 + \cdots$ would sum to 728?
6. The second term of a geometric series is 3 and the common ratio is $\frac{4}{5}$. Find the sum of first 23 consecutive terms in the given geometric series.
7. A geometric series consists of four terms and has a positive common ratio. The sum of the first two terms is 9 and sum of the last two terms is 36. Find the series.
8. Find the sum of first n terms of the series

   (i) $7 + 77 + 777 + \cdots$. (ii) $0.4 + 0.94 + 0.994 + \cdots$.
9. Suppose that five people are ill during the first week of an epidemic and each sick person spreads the contagious disease to four other people by the end of the second week and so on. By the end of 15^{th} week, how many people will be affected by the epidemic?

10. A gardener wanted to reward a boy for his good deeds by giving some mangoes. He gave the boy two choices. He could either have 1000 mangoes at once or he could get 1 mango on the first day, 2 on the second day, 4 on the third day, 8 mangoes on the fourth day and so on for ten days. Which option should the boy choose to get the maximum number of mangoes?

11. A geometric series consists of even number of terms. The sum of all terms is 3 times the sum of odd terms. Find the common ratio.

12. If S_1, S_2 and S_3 are the sum of first n, $2n$ and $3n$ terms of a geometric series respectively, then prove that $S_1(S_3 - S_2) = (S_2 - S_1)^2$.

Remarks

The sum of the first n terms of a geometric series with $a = 1$ and common ratio $x \neq 1$, is given by $1 + x + x^2 + \cdots + x^{n-1} = \dfrac{x^n - 1}{x - 1}, \quad x \neq 1$.

Note that the left hand side of the above equation is a special polynomial in x of degree $n - 1$. This formula will be useful in finding the sum of some series.

2.5.3 Special series $\sum\limits_{k=1}^{n} k$, $\sum\limits_{k=1}^{n} k^2$ and $\sum\limits_{k=1}^{n} k^3$

We have already used the symbol Σ for summation.

Let us list out some examples of finite series represented by sigma notation.

Sl. No.	Notation	Expansion
1.	$\sum\limits_{k=1}^{n} k$ or $\sum\limits_{j=1}^{n} j$	$1 + 2 + 3 + \cdots + n$
2.	$\sum\limits_{n=2}^{6} (n-1)$	$1 + 2 + 3 + 4 + 5$
3.	$\sum\limits_{d=0}^{5} (d+5)$	$5 + 6 + 7 + 8 + 9 + 10$
4.	$\sum\limits_{k=1}^{n} k^2$	$1^2 + 2^2 + 3^3 + \cdots + n^2$
5.	$\sum\limits_{k=1}^{10} 3 = 3\sum\limits_{k=1}^{10} 1$	$3[1 + 1 + \cdots 10 \text{ terms}] = 30.$

We have derived that $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$. This can also be obtained using A.P. with $a = 1$, $d = 1$ and $l = n$ as $S_n = \dfrac{n}{2}(a + l) = \dfrac{n}{2}(1 + n)$.

Hence, using sigma notation we write it as $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

Let us derive the formulae for

$$\text{(i)} \quad \sum_{k=1}^{n}(2k-1), \qquad \text{(ii)} \quad \sum_{k=1}^{n} k^2 \quad \text{and} \quad \text{(iii)} \quad \sum_{k=1}^{n} k^3.$$

Proof:

(i) Let us find $\sum_{k=1}^{n}(2k-1) = 1 + 3 + 5 + \cdots + (2n-1)$.

This is an A.P. consisting of n terms with $a = 1,\ d = 2,\ l = (2n-1)$.

$$\therefore \qquad S_n = \frac{n}{2}(1 + 2n - 1) = n^2 \qquad\qquad \left(S_n = \frac{n}{2}(a + l)\right)$$

Thus, $\sum_{k=1}^{n}(2k-1) = n^2$ (1)

Remarks

1. The formula (1) can also be obtained by the following method

$$\sum_{k=1}^{n}(2k-1) = \sum_{k=1}^{n} 2k - \sum_{k=1}^{n} 1 = 2\left(\sum_{k=1}^{n} k\right) - n = \frac{2(n)(n+1)}{2} - n = n^2.$$

2. From (1), $1 + 3 + 5 + \cdots + l = \left(\frac{l+1}{2}\right)^2$, since $l = 2n - 1 \Longrightarrow n = \frac{l+1}{2}$.

(ii) We know that $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$.

$$\therefore \qquad k^3 - (k-1)^3 = k^2 + k(k-1) + (k-1)^2 \quad (\text{ take } a = k \text{ and } b = k-1)$$

$$\Longrightarrow \qquad k^3 - (k-1)^3 = 3k^2 - 3k + 1 \qquad (2)$$

When $k = 1$, $\quad 1^3 - 0^3 = 3(1)^2 - 3(1) + 1$

When $k = 2$, $\quad 2^3 - 1^3 = 3(2)^2 - 3(2) + 1$

When $k = 3$, $\quad 3^3 - 2^3 = 3(3)^2 - 3(3) + 1$. Continuing this, we have

when $k = n$, $\quad n^3 - (n-1)^3 = 3(n)^2 - 3(n) + 1$.

Adding the above equations corresponding to $k = 1, 2, \cdots, n$ column-wise, we obtain

$$n^3 = 3[1^2 + 2^2 + \cdots + n^2] - 3[1 + 2 + \cdots + n] + n$$

Thus, $3[1^2 + 2^2 + \cdots + n^2] = n^3 + 3[1 + 2 + \cdots + n] - n$

$$3\left[\sum_{k=1}^{n} k^2\right] = n^3 + \frac{3n(n+1)}{2} - n$$

Hence, $$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}. \qquad (3)$$

(iii) $\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + \cdots n^3$

Let us observe the following pattern.

$$1^3 = 1 = (1)^2$$
$$1^3 + 2^3 = 9 = (1+2)^2$$
$$1^3 + 2^3 + 3^3 = 36 = (1+2+3)^2$$
$$1^3 + 2^3 + 3^3 + 4^3 = 100 = (1+2+3+4)^2.$$

Extending this pattern to n terms, we get

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = [1+2+3+\cdots+n]^2$$
$$= \left[\frac{n(n+1)}{2}\right]^2$$

Thus, $$\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2 = \left[\frac{n(n+1)}{2}\right]^2. \qquad (4)$$

(i) The sum of the first n natural numbers, $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

(ii) The sum of the first n odd natural numbers, $\sum_{k=1}^{n} (2k-1) = n^2$.

(iii) The sum of first n odd natural numbers (when the last term l is given) is

$$1 + 3 + 5 + \cdots + l = \left(\frac{l+1}{2}\right)^2.$$

(iv) The sum of squares of first n natural numbers,

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

(v) The sum of cubes of the first n natural numbers,

$$\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2.$$

Example 2.29

Find the sum of the following series

(i) $26 + 27 + 28 + \cdots + 60$ (ii) $1 + 3 + 5 + \cdots$ to 25 terms (iii) $31 + 33 + \cdots + 53$.

Solution

(i) We have $26 + 27 + 28 + \cdots + 60 = (1 + 2 + 3 + \cdots + 60) - (1 + 2 + 3 + \cdots + 25)$

$$= \sum_{1}^{60} n - \sum_{1}^{25} n$$
$$= \frac{60(60+1)}{2} - \frac{25(25+1)}{2}$$
$$= (30 \times 61) - (25 \times 13) = 1830 - 325 = 1505.$$

(ii) Here, $n = 25$

$\therefore \quad 1 + 3 + 5 + \cdots$ to 25 terms $= 25^2$ $\qquad \left(\sum_{k=1}^{n}(2k-1) = n^2 \right)$

$= 625.$

(iii) $31 + 33 + \cdots + 53$

$= (1 + 3 + 5 + \cdots + 53) - (1 + 3 + 5 + \cdots + 29)$

$= \left(\frac{53+1}{2}\right)^2 - \left(\frac{29+1}{2}\right)^2$ $\qquad \left(1 + 3 + 5 + \cdots + l = \left(\frac{l+1}{2}\right)^2 \right)$

$= 27^2 - 15^2 = 504.$

Example 2.30

Find the sum of the following series

(i) $1^2 + 2^2 + 3^2 + \cdots + 25^2$ (ii) $12^2 + 13^2 + 14^2 + \cdots + 35^2$

(iii) $1^2 + 3^2 + 5^2 + \cdots + 51^2$.

Solution

(i) Now, $1^2 + 2^2 + 3^2 + \cdots + 25^2 = \sum_{1}^{25} n^2$

$= \frac{25(25+1)(50+1)}{6}$ $\qquad \left(\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \right)$

$= \frac{(25)(26)(51)}{6}$

$\therefore \quad 1^2 + 2^2 + 3^2 + \cdots + 25^2 = 5525.$

(ii) Now, $12^2 + 13^2 + 14^2 + \cdots + 35^2$

$= (1^2 + 2^2 + 3^2 + \cdots + 35^2) - (1^2 + 2^2 + 3^2 + \cdots + 11^2)$

$= \sum_{1}^{35} n^2 - \sum_{1}^{11} n^2$

$= \frac{35(35+1)(70+1)}{6} - \frac{11(12)(23)}{6}$

$= \frac{(35)(36)(71)}{6} - \frac{(11)(12)(23)}{6}$

$= 14910 - 506 = 14404.$

(iii) Now, $1^2 + 3^2 + 5^2 + \cdots + 51^2$

$= (1^2 + 2^2 + 3^2 + \cdots + 51^2) - (2^2 + 4^2 + 6^2 + \cdots 50^2)$

$= \sum_{1}^{51} n^2 - 2^2[1^2 + 2^2 + 3^2 + \cdots + 25^2]$

$$= \sum_{1}^{51} n^2 - 4\sum_{1}^{25} n^2$$

$$= \frac{51(51+1)(102+1)}{6} - 4 \times \frac{25(25+1)(50+1)}{6}$$

$$= \frac{(51)(52)(103)}{6} - 4 \times \frac{25(26)(51)}{6}$$

$$= 45526 - 22100 = 23426.$$

Example 2.31

Find the sum of the series.

(i) $1^3 + 2^3 + 3^3 + \cdots + 20^3$ (ii) $11^3 + 12^3 + 13^3 + \cdots + 28^3$

Solution

(i) $1^3 + 2^3 + 3^3 + \cdots 20^3 = \sum_{1}^{20} n^3$

$$= \left(\frac{20(20+1)}{2}\right)^2 \qquad \text{using } \sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2.$$

$$= \left(\frac{20 \times 21}{2}\right)^2 = (210)^2 = 44100.$$

(ii) Next we consider $11^3 + 12^3 + \cdots + 28^3$

$$= (1^3 + 2^3 + 3^3 + \cdots + 28^3) - (1^3 + 2^3 + \cdots + 10^3)$$

$$= \sum_{1}^{28} n^3 - \sum_{1}^{10} n^3$$

$$= \left[\frac{28(28+1)}{2}\right]^2 - \left[\frac{10(10+1)}{2}\right]^2$$

$$= 406^2 - 55^2 = (406+55)(406-55)$$

$$= (461)(351) = 161811.$$

Example 2.32

Find the value of k, if $1^3 + 2^3 + 3^3 + \cdots + k^3 = 4356$

Solution Note that k is a positive integer.

Given that $1^3 + 2^3 + 3^3 + \cdots + k^3 = 4356$

$$\Longrightarrow \left(\frac{k(k+1)}{2}\right)^2 = 4356 = 6 \times 6 \times 11 \times 11$$

Taking square root, we get $\frac{k(k+1)}{2} = 66$

$$\Longrightarrow k^2 + k - 132 = 0 \Longrightarrow (k+12)(k-11) = 0$$

Thus, $k = 11$, since k is positive.

Example 2.33

(i) If $1 + 2 + 3 + \cdots + n = 120$, find $1^3 + 2^3 + 3^3 + \cdots n^3$.

(ii) If $1^3 + 2^3 + 3^3 + \cdots + n^3 = 36100$, then find $1 + 2 + 3 + \cdots + n$.

Solution

(i) Given $1 + 2 + 3 + \cdots + n = 120$ i.e. $\frac{n(n+1)}{2} = 120$

$$\therefore \quad 1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2 = 120^2 = 14400$$

(ii) Given $1^3 + 2^3 + 3^3 + \cdots + n^3 = 36100$

$$\Longrightarrow \quad \left(\frac{n(n+1)}{2}\right)^2 = 36100 = 19 \times 19 \times 10 \times 10$$

$$\Longrightarrow \quad \frac{n(n+1)}{2} = 190$$

Thus, $1 + 2 + 3 + \cdots + n = 190$.

Example 2.34

Find the total area of 14 squares whose sides are 11 cm, 12 cm, $\cdots$, 24 cm, respectively.

Solution The areas of the squares form the series $11^2 + 12^2 + \cdots + 24^2$

$$\begin{aligned}
\text{Total area of 14 squares } &= 11^2 + 12^2 + 13^2 + \cdots + 24^2 \\
&= (1^2 + 2^2 + 3^2 + \cdots + 24^2) - (1^2 + 2^2 + 3^2 + \cdots + 10^2) \\
&= \sum_1^{24} n^2 - \sum_1^{10} n^2 \\
&= \frac{24(24+1)(48+1)}{6} - \frac{10(10+1)(20+1)}{6} \\
&= \frac{(24)(25)(49)}{6} - \frac{(10)(11)(21)}{6} \\
&= 4900 - 385 \\
&= 4515 \text{ sq. cm.}
\end{aligned}$$

Exercise 2.6

1. Find the sum of the following series.

(i) $1 + 2 + 3 + \cdots + 45$ (ii) $16^2 + 17^2 + 18^2 + \cdots + 25^2$

(iii) $2 + 4 + 6 + \cdots + 100$ (iv) $7 + 14 + 21 \cdots + 490$

(v) $5^2 + 7^2 + 9^2 + \cdots + 39^2$ (vi) $16^3 + 17^3 + \cdots + 35^3$

2. Find the value of k if

(i) $1^3 + 2^3 + 3^3 + \cdots + k^3 = 6084$ (ii) $1^3 + 2^3 + 3^3 + \cdots + k^3 = 2025$

3. If $1 + 2 + 3 + \cdots + p = 171$, then find $1^3 + 2^3 + 3^3 + \cdots + p^3$.

4. If $1^3 + 2^3 + 3^3 + \cdots + k^3 = 8281$, then find $1 + 2 + 3 + \cdots + k$.

5. Find the total area of 12 squares whose sides are 12cm, 13cm, $\cdots$, 23cm. respectively.

6. Find the total volume of 15 cubes whose edges are 16cm, 17cm, 18cm, $\cdots$, 30cm respectively.

Exercise 2.7

Choose the correct answer.

1. Which one of the following is not true?

(A) A sequence is a real valued function defined on $\mathbb{N}$.

(B) Every function represents a sequence.

(C) A sequence may have infinitely many terms.

(D) A sequence may have a finite number of terms.

2. The 8th term of the sequence 1, 1, 2, 3, 5, 8, $\cdots$ is

(A) 25 (B) 24 (C) 23 (D) 21

3. The next term of $\frac{1}{20}$ in the sequence $\frac{1}{2}, \frac{1}{6}, \frac{1}{12}, \frac{1}{20}, \cdots$ is

(A) $\frac{1}{24}$ (B) $\frac{1}{22}$ (C) $\frac{1}{30}$ (D) $\frac{1}{18}$

4. If a, b, c, l, m are in A.P, then the value of $a - 4b + 6c - 4l + m$ is

(A) 1 (B) 2 (C) 3 (D) 0

5. If a, b, c are in A.P. then $\frac{a-b}{b-c}$ is equal to

(A) $\frac{a}{b}$ (B) $\frac{b}{c}$ (C) $\frac{a}{c}$ (D) 1

6. If the n^{th} term of a sequence is $100n+10$, then the sequence is

(A) an A.P. (B) a G.P. (C) a constant sequence (D) neither A.P. nor G.P.

7. If $a_1, a_2, a_3, \cdots$ are in A.P. such that $\frac{a_4}{a_7} = \frac{3}{2}$, then the 13th term of the A.P. is

(A) $\frac{3}{2}$ (B) 0 (C) $12a_1$ (D) $14a_1$

8. If the sequence $a_1, a_2, a_3, \cdots$ is in A.P. , then the sequence $a_5, a_{10}, a_{15}, \cdots$ is

(A) a G.P. (B) an A.P. (C) neither A.P nor G.P. (D) a constant sequence

9. If $k+2$, $4k-6$, $3k-2$ are the three consecutive terms of an A.P, then the value of k is

(A) 2 (B) 3 (C) 4 (D) 5

10. If $a, b, c, l, m. n$ are in A.P., then $3a+7, 3b+7, 3c+7, 3l+7, 3m+7, 3n+7$ form

(A) a G.P. (B) an A.P. (C) a constant sequence (D) neither A.P. nor G.P

11. If the third term of a G.P is 2, then the product of first 5 terms is

(A) 5^2 (B) 2^5 (C) 10 (D) 15

12. If a, b, c are in G.P, then $\frac{a-b}{b-c}$ is equal to

(A) $\frac{a}{b}$ (B) $\frac{b}{a}$ (C) $\frac{a}{c}$ (D) $\frac{c}{b}$

13. If $x,\ 2x+2,\ 3x+3$ are in G.P, then $5x,\ 10x+10,\ 15x+15$ form

(A) an A.P. (B) a G.P. (C) a constant sequence (D) neither A.P. nor a G.P.

14. The sequence –3, –3, –3,$\cdots$ is

(A) an A.P. only (B) a G.P. only (C) neither A.P. nor G.P (D) both A.P. and G.P.

15. If the product of the first four consecutive terms of a G.P is 256 and if the common ratio is 4 and the first term is positive, then its 3rd term is

(A) 8 (B) $\frac{1}{16}$ (C) $\frac{1}{32}$ (D) 16

16. In a G.P, $t_2 = \frac{3}{5}$ and $t_3 = \frac{1}{5}$. Then the common ratio is

(A) $\frac{1}{5}$ (B) $\frac{1}{3}$ (C) 1 (D) 5

17. If $x \neq 0$, then $1 + \sec x + \sec^2 x + \sec^3 x + \sec^4 x + \sec^5 x$ is equal to

(A) $(1+\sec x)(\sec^2 x + \sec^3 x + \sec^4 x)$ (B) $(1+\sec x)(1+\sec^2 x + \sec^4 x)$

(C) $(1-\sec x)(\sec x + \sec^3 x + \sec^5 x)$ (D) $(1+\sec x)(1+\sec^3 x + \sec^4 x)$

18. If the n^{th} term of an A.P. is $t_n = 3 - 5n$, then the sum of the first n terms is

(A) $\frac{n}{2}[1-5n]$ (B) $n(1-5n)$ (C) $\frac{n}{2}(1+5n)$ (D) $\frac{n}{2}(1+n)$

19. The common ratio of the G.P. a^{m-n} , a^m, a^{m+n} is

(A) a^m (B) a^{-m} (C) a^n (D) a^{-n}

20. If 1 + 2 + 3 +. . . + $n = k$ then $1^3 + 2^3 + \cdots + n^3$ is equal to

(A) k^2 (B) k^3 (C) $\frac{k(k+1)}{2}$ (D) $(k+1)^3$

Points to Remember

- A sequence of real numbers is an arrangement or a list of real numbers in a specific order.
- The sequence given by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$, $n = 3, 4, \cdots$ is called the Fibonacci sequence which is nothing but 1, 1, 2, 3, 5, 8, 13, 21, 34, $\cdots$
- A sequence $a_1, a_2, a_3, \cdots, a_n, \cdots$ is called an arithmetic sequence if $a_{n+1} = a_n + d$, $n \in \mathbb{N}$ where d is a constant. Here a_1 is called the first term and the constant d is called the common difference.

  The formula for the general term of an A.P. is $t_n = a + (n-1)d \quad \forall\ n \in \mathbb{N}$.
- A sequence $a_1, a_2, a_3, \cdots, a_n, \cdots$ is called a geometric sequence if $a_{n+1} = a_n r$, where $r \neq 0$, $n \in \mathbb{N}$ where r is a constant. Here, a_1 is the first term and the constant r is called the common ratio. The formula for the general term of a G.P. is $t_n = ar^{n-1}$, $n = 1, 2, 3, \cdots$.
- An expression of addition of terms of a sequence is called a series. If the sum consists only finite number of terms, then it is called a finite series. If the sum consists of infinite number of terms of a sequence, then it is called an infinite series.
- The sum S_n of the first n terms of an arithmetic sequence with first term a and common difference d is given by $S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}(a + l)$, where l is the last term.
- The sum of the first n terms of a geometric series is given by

  $$S_n = \begin{cases} \dfrac{a(r^n - 1)}{r - 1} = \dfrac{a(1 - r^n)}{1 - r}, & \text{if } r \neq 1 \\ na & \text{if } r = 1. \end{cases}$$

  where a is the first term and r is the common ratio.
- The sum of the first n natural numbers, $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.
- The sum of the first n odd natural numbers, $\sum_{k=1}^{n} (2k-1) = n^2$
- The sum of first n odd natural numbers (when the last term l is given) is

  $$1 + 3 + 5 + \cdots + l = \left(\frac{l+1}{2}\right)^2.$$
- The sum of squares of first n natural numbers, $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.
- The sum of cubes of the first n natural numbers, $\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2$.

Do you know?

A Mersenne number, named after Marin Mersenne, is a positive integer of the form $M = 2^p - 1$, where p is a positive integer. If M is a prime, then it is called a Mersenne prime.Interestingly, if $2^p - 1$ is prime, then p is prime.The largest known prime number $2^{43,112,609} - 1$ is a Mersenne prime.

3 ALGEBRA

The human mind has never invented a labour-saving machine equal to algebra - Author unknown

- Introduction
- Polynomials
- Synthetic Division
- GCD and LCM
- Rational Expressions
- Square root
- Quadratic Equations

Al-Khwarizmi

(780-850)

Arab

Al-Khwarizmi's contribution to Mathematics and Geography established the basis for innovation in Algebra and Trigonometry. He presented the first systematic solution of linear and quadratic equations.

He is considered the founder of algebra. His work on arithmetic was responsible for introducing the Arabic numerals based on the Hindu-Arabic numeral system developed in Indian Mathematics, to the Western world.

3.1 Introduction

Algebra is an important and a very old branch of mathematics which deals with solving algebraic equations. In third century, the Greek mathematician Diophantus wrote a book "Arithmetic" which contained a large number of practical problems. In the sixth and seventh centuries, Indian mathematicians like Aryabhatta and Brahmagupta have worked on linear equations and quadratic equations and developed general methods of solving them.

The next major development in algebra took place in ninth century by Arab mathematicians. In particular, Al-Khwarizmi's book entitled "Compendium on calculation by completion and balancing" was an important milestone. There he used the word aljabra - which was latinized into algebra - translates as competition or restoration. In the 13th century, Leonardo Fibonacci's books on algebra was important and influential. Other highly influential works on algebra were those of the Italian mathematician Luca Pacioli (1445-1517), and of the English mathematician Robert Recorde (1510-1558).

In later centuries Algebra blossomed into more abstract and in 19th century British mathematicians took the lead in this effort. Peacock (Britain, 1791-1858) was the founder of axiomatic thinking in arithmetic and algebra. For this reason he is sometimes called the "Euclid of Algebra". DeMorgan (Britain, 1806-1871) extended Peacock's work to consider operations defined on abstract symbols.

In this chapter, we shall focus on learning techniques of solving linear system of equations and quadratic equations.

3.2 System of linear equations in two unknowns

In class IX, we have studied the linear equation $ax + b = 0,\ a \neq 0$, in one unknown x.

Let us consider a linear equation $ax + by = c$, where at least one of a and b is non-zero, in two unknowns x and y. An ordered pair (x_0, y_0) is called a solution to the linear equation if the values $x = x_0,\ y = y_0$ satisfy the equation.

Geometrically, the graph of the linear equation $ax + by = c$ is a straight line in a plane. So each point (x, y) on this line corresponds to a solution of the equation $ax + by = c$. Conversely, every solution (x, y) of the equation is a point on this straight line. Thus, the equation $ax + by = c$ has infinitely many solutions.

A set of finite number of linear equations in two unknowns x and y that are to be treated together, is called a system of linear equations in x and y. Such a system of equations is also called simultaneous equations.

Definition

An ordered pair (x_0, y_0) is called a solution to a linear system in two variables if the values $x = x_0,\ y = y_0$ satisfy all the equations in the system.

A system of linear equations

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

in two variables is said to be

(i) consistent if at least one pair of values of x and y satisfies both equations and

(ii) inconsistent if there are no values of x and y that satisfy both equations.

In this section, we shall discuss only a pair of linear equations in two variables.

Remarks

(i) An equation of the form $ax + by = c$ is called linear because the variables are only to the first power, and there are no products of variables in the equation.

(ii) It is also possible to consider linear systems in more than two variables. You will learn this in higher classes.

Let us consider a linear system

$$a_1x + b_1y = c_1 \qquad (1)$$
$$a_2x + b_2y = c_2 \qquad (2)$$

in two variables x and y, where any of the constants a_1, b_1, a_2 and b_2 can be zero with the exception that each equation must have at least one variable in it or simply,

$$a_1^2 + b_1^2 \neq 0\ ,\ a_2^2 + b_2^2 \neq 0.$$

Geometrically the following situations occur. The two straight lines represented by (1) and (2)

(i) may intersect at exactly one point

(ii) may not intersect at any point

(iii) may coincide.

If (i) happens, then the intersecting point gives the unique solution of the system. If (ii) happens, then the system does not have a solution. If (iii) happens, then every point on the line corresponds to a solution to the system. Thus, the system will have infinitely many solutions in this case.

Now, we will solve a system of linear equations in two unknowns using the following algebraic methods (i) the method of elimination (ii) the method of cross multiplication.

3.2.1 Elimination method

In this method, we may combine equations of a system in such a manner as to get rid of one of the unknowns. The elimination of one unknown can be achieved in the following ways.

(i) Multiply or divide the members of the equations by such numbers as to make the coefficients of the unknown to be eliminated numerically equal.

(ii) Then, eliminate by addition if the resulting coefficients have unlike signs and by subtraction if they have like signs.

Example 3.1

Solve $3x - 5y = -16$, $2x + 5y = 31$

Solution The given equations are

$$3x - 5y = -16 \qquad (1)$$

$$2x + 5y = 31 \qquad (2)$$

Note that the coefficients of y in both equations are numerically equal.

So, we can eliminate y easily.

Adding (1) and (2), we obtain an equation

$$5x = 15 \qquad (3)$$

That is, $x = 3$.

Now, we substitute $x = 3$ in (1) or (2) to solve for y.

Substituting $x = 3$ in (1) we obtain, $3(3) - 5y = -16$

$$\Longrightarrow \quad y = 5.$$

Now, (3, 5) a is solution to the given system because (1) and (2) are true when $x = 3$ and $y = 5$ as from (1) and (2) we get, $3(3) - 5(5) = -16$ and $2(3) + 5(5) = 31$.

Note

Obtaining equation (3) in only one variable is an important step in finding the solution. We obtained equation (3) in one variable x by eliminating the variable y. So this method of solving a system by eliminating one of the variables first, is called "method of elimination".

Example 3.2

The cost of 11 pencils and 3 erasers is ₹ 50 and the cost of 8 pencils and 3 erasers is ₹ 38. Find the cost of each pencil and each eraser.

Solution Let x denote the cost of a pencil in rupees and y denote the cost of an eraser in rupees.

Then according to the given information we have

$$11x + 3y = 50 \quad (1)$$

$$8x + 3y = 38 \quad (2)$$

Subtracting (2) from (1) we get, $3x = 12$ which gives $x = 4$.

Now substitute $x = 4$ in (1) to find the value of y. We get,

$$11(4) + 3y = 50 \qquad \text{i.e.,} \quad y = 2.$$

Therefore, $x = 4$ and $y = 2$ is the solution of the given pair of equations.

Thus, the cost of a pencil is ₹ 4 and that of an eraser is ₹ 2.

Note

It is always better to check that the obtained values satisfy the both equations.

Example 3.3

Solve by elimination method $3x + 4y = -25$, $2x - 3y = 6$

Solution The given system is

$$3x + 4y = -25 \quad (1)$$

$$2x - 3y = 6 \quad (2)$$

To eliminate the variable x, let us multiply (1) by 2 and (2) by –3 to obtain

$$(1) \times 2 \implies 6x + 8y = -50 \quad (3)$$

$$(2) \times -3 \implies -6x + 9y = -18 \quad (4)$$

Now, adding (3) and (4) we get, $17y = -68$ which gives $y = -4$

Next, substitute $y = -4$ in (1) to obtain

$$3x + 4(-4) = -25$$

That is, $x = -3$

Hence, the solution is $(-3, -4)$.

Remarks

In Example 3.3, it is not possible to eliminate one of the variables by simply adding or subtracting the given equations as we did in Example 3.1. Thus, first we shall do some manipulations so that coefficients of either x or y are equal except for sign. Then we do the elimination.

Example 3.4

Using elimination method, solve $101x + 99y = 499$, $99x + 101y = 501$

Solution The given system of equations is

$$101x + 99y = 499 \qquad (1)$$

$$99x + 101y = 501 \qquad (2)$$

Here, of course we could multiply equations by appropriate numbers to eliminate one of the variables.

However, note that the coefficient of x in one equation is equal to the coefficient of y in the other equation. In such a case, we add and subtract the two equations to get a new system of very simple equations having the same solution.

Adding (1) and (2), we get $200x + 200y = 1000$.

Dividing by 200 we get, $x + y = 5 \qquad (3)$

Subtracting (2) from (1), we get $2x - 2y = -2$ which is same as

$$x - y = -1 \qquad (4)$$

Solving (3) and (4), we get $x = 2$, $y = 3$.

Thus, the required solution is (2, 3).

Example 3.5

Solve $3(2x + y) = 7xy$; $3(x + 3y) = 11xy$ using elimination method

Solution The given system of equations is

$$3(2x + y) = 7xy \qquad (1)$$

$$3(x + 3y) = 11xy \qquad (2)$$

Observe that the given system is not linear because of the occurrence of xy term.

Also, note that if $x = 0$, then $y = 0$ and vice versa. So, (0, 0) is a solution for the system and any other solution would have both $x \neq 0$ and $y \neq 0$.

Thus, we consider the case where $x \neq 0$, $y \neq 0$.

Dividing both sides of each equation by xy, we get

$$\frac{6}{y} + \frac{3}{x} = 7, \quad \text{i.e.,} \quad \frac{3}{x} + \frac{6}{y} = 7 \qquad (3)$$

and

$$\frac{9}{x} + \frac{3}{y} = 11 \qquad (4)$$

Let $a = \frac{1}{x}$ and $b = \frac{1}{y}$.

Equations (3) and (4) become

$$3a + 6b = 7 \qquad (5)$$

$$9a + 3b = 11 \qquad (6)$$

which is a linear system in a and b.

To eliminate b, we have (6) $\times 2 \Longrightarrow \quad 18a + 6b = 22$ (7)

Subtracting (7) from (5) we get, $\quad -15a = -15$. That is, $a = 1$.

Substituting $a = 1$ in (5) we get, $\quad b = \frac{2}{3}$. Thus, $a = 1$ and $b = \frac{2}{3}$.

When $a = 1$, we have $\frac{1}{x} = 1$. $\quad$ Thus, $x = 1$.

When $b = \frac{2}{3}$, we have $\frac{1}{y} = \frac{2}{3}$. $\quad$ Thus, $y = \frac{3}{2}$.

Thus, the system has two solutions $\left(1, \frac{3}{2}\right)$ and $(0, 0)$.

Aliter

The given system of equations can also be solved in the following way.

Now, $\quad 3(2x + y) = 7xy \qquad (1)$

$\quad 3(x + 3y) = 11xy \qquad (2)$

Now, (2) × 2 – (1) $\Longrightarrow \quad 15y = 15xy$

$\Longrightarrow \quad 15y(1-x) = 0$. Thus, $x = 1$ and $y = 0$

When $x = 1$, we have $y = \frac{3}{2}$ and when $y = 0$, we have $x = 0$

Hence, the two solutions are $\left(1, \frac{3}{2}\right)$ and $(0, 0)$.

Note : In $15y = 15xy$, y is not to be cancelled out as $y = 0$ gives another solution.

Exercise 3.1

Solve each of the following system of equations by elimination method.

1. $x + 2y = 7, \quad x - 2y = 1$
2. $3x + y = 8, \quad 5x + y = 10$
3. $x + \frac{y}{2} = 4, \quad \frac{x}{3} + 2y = 5$
4. $11x - 7y = xy, \quad 9x - 4y = 6xy$
5. $\frac{3}{x} + \frac{5}{y} = \frac{20}{xy}, \frac{2}{x} + \frac{5}{y} = \frac{15}{xy}, x \neq 0, y \neq 0$
6. $8x - 3y = 5xy, \quad 6x - 5y = -2xy$
7. $13x + 11y = 70, \quad 11x + 13y = 74$
8. $65x - 33y = 97, \quad 33x - 65y = 1$
9. $\frac{15}{x} + \frac{2}{y} = 17, \frac{1}{x} + \frac{1}{y} = \frac{36}{5}, x \neq 0, y \neq 0$
10. $\frac{2}{x} + \frac{2}{3y} = \frac{1}{6}, \frac{3}{x} + \frac{2}{y} = 0, x \neq 0, y \neq 0$

Cardinality of the set of solutions of the system of linear equations

Let us consider the system of two equations

$$a_1x + b_1y + c_1 = 0 \qquad (1)$$

$$a_2x + b_2y + c_2 = 0 \qquad (2)$$

where the coefficients are real numbers such that $a_1^2 + b_1^2 \neq 0$, $a_2^2 + b_2^2 \neq 0$.

Let us apply the elimination method for equating the coefficients of y.

Now, multiply equation (1) by b_2 and equation (2) by b_1, we get,

$$b_2a_1x + b_2b_1y + b_2c_1 = 0 \quad (3)$$

$$b_1a_2x + b_1b_2y + b_1c_2 = 0 \quad (4)$$

Subtracting equation (4) from (3), we get

$$(b_2a_1 - b_1a_2)x = b_1c_2 - b_2c_1 \Longrightarrow x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1} \quad \text{provided } a_1b_2 - a_2b_1 \neq 0$$

Substituting the value of x in either (1) or (2) and solving for y, we get

$$y = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}, \quad \text{provided } a_1b_2 - a_2b_1 \neq 0.$$

Thus, we have

$$x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1} \text{ and } y = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}, \; a_1b_2 - a_2b_1 \neq 0. \quad (5)$$

Here, we have to consider two cases.

***Case* (i)** $a_1b_2 - a_2b_1 \neq 0$. That is, $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$.

In this case, the pair of linear equations has a unique solution.

***Case* (ii)** $a_1b_2 - a_2b_1 = 0$. That is, $\frac{a_1}{a_2} = \frac{b_1}{b_2}$ if $a_2 \neq 0$ and $b_2 \neq 0$.

In this case, let $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \lambda$. Then $a_1 = \lambda a_2$, $b_1 = \lambda b_2$

Now, substituting the values of a_1 and b_1 in equation (1) we get,

$$\lambda(a_2x + b_2y) + c_1 = 0 \quad (6)$$

It is easily observed that both the equations (6) and (2) can be satisfied only if

$$c_1 = \lambda c_2 \Longrightarrow \frac{c_1}{c_2} = \lambda$$

If $c_1 = \lambda c_2$, any solution of equation (2) will also satisfy the equation (1) and vice versa.

So, if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \lambda$; then there are infinitely many solutions to the pair of linear equations given by (1) and (2).

If $c_1 \neq \lambda c_2$, then any solution of equation (1) will not satisfy equation (2) and vice versa.

Hence, if $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$, then the pair of linear equations given by (1) and (2) has no solution.

Note

Now, we summarise the above discussion.

For the system of equations

$$a_1x + b_1y + c_1 = 0$$
$$a_2x + b_2y + c_2 = 0, \text{ where } a_1^2 + b_1^2 \neq 0 \;,\; a_2^2 + b_2^2 \neq 0.$$

(i) If $a_1b_2 - b_1a_2 \neq 0$ or $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$, then the system of equations has a unique solution.

(ii) If $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$, then the system of equations has infinitely many solutions.

(iii) If $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$, then the system of equations has no solution.

3.2.2 Cross multiplication method

While solving a pair of linear equations in two unknowns x and y using elimination method, we utilised the coefficients effectively to get the solution. There is another method called the cross multiplication method, which simplifies the procedure. Now, let us describe this method and see how it works.

Let us consider the system

$$a_1x + b_1y + c_1 = 0 \qquad (1)$$

$$a_2x + b_2y + c_2 = 0 \text{ with } a_1b_2 - b_1a_2 \neq 0 \qquad (2)$$

We have already established that the system has the solution

$$x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \quad y = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$$

Thus, we can write $\dfrac{x}{b_1c_2 - b_2c_1} = \dfrac{1}{a_1b_2 - a_2b_1}, \quad \dfrac{y}{c_1a_2 - c_2a_1} = \dfrac{1}{a_1b_2 - a_2b_1}$

Let us write the above in the following form

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1}.$$

The following arrow diagram may be very useful in remembering the above relation.

$$\begin{array}{ccccccc} & & x & & y & & 1 \\ b_1 & & c_1 & & a_1 & & b_1 \\ b_2 & & c_2 & & a_2 & & b_2 \end{array}$$

The arrows between the two numbers indicate that they are multiplied, the second product (upward arrow) is to be subtracted from the first product (downward arrow).

Method of solving a linear system of equations by the above form is called the cross multiplication method.

Note that in the representation $\dfrac{x}{b_1c_2 - b_2c_1} = \dfrac{y}{c_1a_2 - c_2a_1} = \dfrac{1}{a_1b_2 - a_2b_1}$,

$b_1c_2 - b_2c_1$ or $c_1a_2 - c_2a_1$ may be equal to 0 but $a_1b_2 - a_2b_1 \neq 0$.

> Hence, for the system of equations $\begin{aligned} a_1x + b_1y + c_1 &= 0 \\ a_2x + b_2y + c_2 &= 0 \end{aligned}$
>
> (i) if $b_1c_2 - b_2c_1 = 0$ and $a_1b_2 - a_2b_1 \neq 0$, then $x = 0$
>
> (ii) if $c_1a_2 - c_2a_1 = 0$ and $a_1b_2 - a_2b_1 \neq 0$, then $y = 0$

Hereafter, we shall mostly restrict ourselves to the system of linear equations having unique solution and find the solution by the method of cross multiplication.

Example 3.6

Solve

$$\begin{aligned} 2x + 7y - 5 &= 0 \\ -3x + 8y &= -11 \end{aligned}$$

Solution The given system of equations is

$$\begin{aligned} 2x + 7y - 5 &= 0 \\ -3x + 8y + 11 &= 0 \end{aligned}$$

For the cross multiplication method, we write the coefficients as

$$\begin{array}{cccccc} & x & & y & & 1 \\ 7 & & -5 & & 2 & & 7 \\ 8 & & 11 & & -3 & & 8 \end{array}$$

Hence, we get $\dfrac{x}{(7)(11)-(8)(-5)} = \dfrac{y}{(-5)(-3)-(2)(11)} = \dfrac{1}{(2)(8)-(-3)(7)}$.

That is, $\dfrac{x}{117} = \dfrac{y}{-7} = \dfrac{1}{37}$. i.e., $x = \dfrac{117}{37}$, $y = -\dfrac{7}{37}$.

Hence, the solution is $\left(\dfrac{117}{37}, -\dfrac{7}{37}\right)$.

Example 3.7

Using cross multiplication method, solve $\begin{aligned} 3x + 5y &= 25 \\ 7x + 6y &= 30 \end{aligned}$

Solution The given system of equations is $\begin{aligned} 3x + 5y - 25 &= 0 \\ 7x + 6y - 30 &= 0 \end{aligned}$

Now, writing the coefficients for cross multiplication, we get

$$\begin{array}{cccccc} & x & & y & & 1 \\ 5 & & -25 & & 3 & & 5 \\ 6 & & -30 & & 7 & & 6 \end{array}$$

$$\Longrightarrow \quad \frac{x}{-150+150} = \frac{y}{-175+90} = \frac{1}{18-35}. \text{ i.e., } \frac{x}{0} = \frac{y}{-85} = \frac{1}{-17}.$$

Thus, we have $x = 0$, $y = 5$. Hence, the solution is (0, 5).

Note

Here, $\frac{x}{0} = -\frac{1}{17}$ is to mean $x = \frac{0}{-17} = 0$. Thus $\frac{x}{0}$ is only a notation and it is not division by zero. It is always true that division by zero is not defined.

Example 3.8

In a two digit number, the digit in the unit place is twice of the digit in the tenth place. If the digits are reversed, the new number is 27 more than the given number. Find the number.

Solution Let x denote the digit in the tenth place and y denote the digit in unit place.. So, the number may be written as $10x + y$ in the expanded form. (just like 35= 10(3) +5)

When the digits are reversed, x becomes the digit in unit place and y becomes the digit in the tenth place. The changed number, in the expanded form is $10y + x$.

According to the first condition, we have $y = 2x$ which is written as

$$2x - y = 0 \qquad (1)$$

Also, by second condition, we have

$$(10y + x) - (10x + y) = 27$$

That is, $-9x + 9y = 27 \Longrightarrow -x + y = 3 \qquad (2)$

Adding equations (1) and (2), we get $x = 3$.

Substituting $x = 3$ in the equation (2), we get $y = 6$.

Thus, the given number is $(3\times10) + 6 = 36$.

Example 3.9

A fraction is such that if the numerator is multiplied by 3 and the denominator is reduced by 3, we get $\frac{18}{11}$, but if the numerator is increased by 8 and the denominator is doubled, we get $\frac{2}{5}$. Find the fraction.

Solution Let the fraction be $\frac{x}{y}$. According to the given conditions, we have

$$\frac{3x}{y-3} = \frac{18}{11} \quad \text{and} \quad \frac{x+8}{2y} = \frac{2}{5}$$

$$\Longrightarrow \quad 11x = 6y - 18 \quad \text{and} \quad 5x + 40 = 4y$$

So, we have $\quad 11x - 6y + 18 = 0 \qquad (1)$

$\quad 5x - 4y + 40 = 0 \qquad (2)$

On comparing the coefficients of (1) and (2) with $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$,

we have $\quad a_1 = 11,\ b_1 = -6,\ c_1 = 18\ ;\quad a_2 = 5,\quad b_2 = -4,\quad c_2 = 40.$

Thus, $\quad a_1b_2 - a_2b_1 = (11)(-4) - (5)(-6) = -14 \neq 0.$

Hence, the system has a unique solution.

Now, writing the coefficients for the cross multiplication, we have

$$\begin{array}{cccccc} & x & & y & & 1 \\ -6 & & 18 & & 11 & & -6 \\ -4 & & 40 & & 5 & & -4 \end{array}$$

$$\Longrightarrow \quad \frac{x}{-240 + 72} = \frac{y}{90 - 440} = \frac{1}{-44 + 30}$$

$$\Longrightarrow \quad \frac{x}{-168} = \frac{y}{-350} = \frac{1}{-14}$$

Thus, $\quad x = \dfrac{168}{14} = 12\ ;\quad y = \dfrac{350}{14} = 25.$ Hence, the fraction is $\dfrac{12}{25}$.

Example 3.10

Eight men and twelve boys can finish a piece of work in 10 days while six men and eight boys can finish the same work in 14 days. Find the number of days taken by one man alone to complete the work and also one boy alone to complete the work.

Solution Let x denote the number of days needed for one man to finish the work and y denote the number of days needed for one boy to finish the work. Clearly, $x \neq 0$ and $y \neq 0$.

So, one man can complete $\dfrac{1}{x}$ part of the work in one day and one boy can complete $\dfrac{1}{y}$ part of the work in one day.

The amount of work done by 8 men and 12 boys in one day is $\dfrac{1}{10}$.

Thus, we have $\quad \dfrac{8}{x} + \dfrac{12}{y} = \dfrac{1}{10} \qquad (1)$

The amount of work done by 6 men and 8 boys in one day is $\dfrac{1}{14}$.

Thus, we have $\quad \dfrac{6}{x} + \dfrac{8}{y} = \dfrac{1}{14} \qquad (2)$

Let $\quad a = \dfrac{1}{x}$ and $b = \dfrac{1}{y}$. Then (1) and (2) give, respectively,

$$8a + 12b = \frac{1}{10} \quad \Longrightarrow \quad 4a + 6b - \frac{1}{20} = 0. \qquad (3)$$

$$6a + 8b = \frac{1}{14} \quad \Longrightarrow \quad 3a + 4b - \frac{1}{28} = 0. \qquad (4)$$

Writing the coefficients of (3) and (4) for the cross multiplication, we have

$$\begin{array}{cccc} & a & b & 1 \\ 6 & -\dfrac{1}{20} & 4 & 6 \\ 4 & -\dfrac{1}{28} & 3 & 4 \end{array}$$

Thus, we have $\dfrac{a}{-\dfrac{3}{14}+\dfrac{1}{5}} = \dfrac{b}{-\dfrac{3}{20}+\dfrac{1}{7}} = \dfrac{1}{16-18}$. i.e., $\dfrac{a}{-\dfrac{1}{70}} = \dfrac{b}{-\dfrac{1}{140}} = \dfrac{1}{-2}$.

That is, $a = \dfrac{1}{140}$, $b = \dfrac{1}{280}$

Thus, we have $x = \dfrac{1}{a} = 140$, $y = \dfrac{1}{b} = 280$.

Hence, one man can finish the work individually in 140 days and one boy can finish the work individually in 280 days.

Exercise 3.2

1. Solve the following systems of equations using cross multiplication method.

   (i) $3x + 4y = 24,\ 20x - 11y = 47$ (ii) $0.5x + 0.8y = 0.44,\ 0.8x + 0.6y = 0.5$

   (iii) $\dfrac{3x}{2} - \dfrac{5y}{3} = -2,\ \dfrac{x}{3} + \dfrac{y}{2} = \dfrac{13}{6}$ (iv) $\dfrac{5}{x} - \dfrac{4}{y} = -2,\ \dfrac{2}{x} + \dfrac{3}{y} = 13$

2. Formulate the following problems as a pair of equations, and hence find their solutions:

   (i) One number is greater than thrice the other number by 2. If 4 times the smaller number exceeds the greater by 5, find the numbers.

   (ii) The ratio of income of two persons is 9 : 7 and the ratio of their expenditure is 4 : 3. If each of them manages to save ₹ 2000 per month, find their monthly income.

   (iii) A two digit number is seven times the sum of its digits. The number formed by reversing the digits is 18 less than the given number. Find the given number.

   (iv) Three chairs and two tables cost ₹ 700 and five chairs and three tables cost ₹ 1100. What is the total cost of 2 chairs and 3 tables?

   (v) In a rectangle, if the length is increased and the breadth is reduced each by 2 cm then the area is reduced by 28 cm^2. If the length is reduced by 1 cm and the breadth increased by 2cm , then the area increases by 33cm^2. Find the area of the rectangle.

   (vi) A train travelled a certain distance at a uniform speed. If the train had been 6 km/hr faster, it would have taken 4 hours less than the scheduled time. If the train were slower by 6 km/hr, then it would have taken 6 hours more than the scheduled time. Find the distance covered by the train.

3.3 Quadratic polynomials

A polynomial of degree n in the variable x is $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n$ where $a_0 \neq 0$ and $a_1, a_2, a_3, ..., a_n$ are real constants.

A polynomial of degree two is called a quadratic polynomial and is normally written as $p(x) = ax^2 + bx + c$, where $a \neq 0$, b and c are real constants. Real constants are polynomials of degree zero.

For example, $x^2 + x + 1$, $3x^2 - 1$, $-\frac{3}{2}x^2 + 2x - \frac{7}{3}$ are quadratic polynomials.

The value of a quadratic polynomial $p(x) = ax^2 + bx + c$ at $x = k$ is obtained by replacing x by k in $p(x)$. Thus, the value of $p(x)$ at $x = k$ is $p(k) = ak^2 + bk + c$.

3.3.1 Zeros of a polynomial

Consider a polynomial $p(x)$. If k is a real number such that $p(k) = 0$, then k is called a zero of the polynomial $p(x)$.

For example,

the zeros of the polynomial $q(x) = x^2 - 5x + 6$ are 2 and 3 because $q(2) = 0$ and $q(3) = 0$.

Remarks

A polynomial may not have any zero in real numbers at all. For example, $p(x) = x^2 + 1$ has no zeros in real numbers. That is, there is no real k such that $p(k) = 0$. Geometrically a zero of any polynomial is nothing but the x-coordinate of the point of intersection of the graph of the polynomial and the x-axis if they intersect. (see Fig. 3.1 and Fig. 3.2)

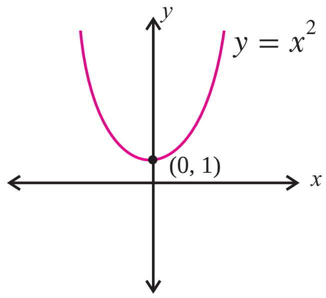


Fig. 3.1

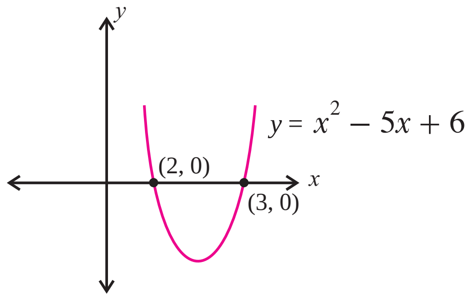


Fig. 3.2

3.3.2 Relationship between zeros and coefficients of a quadratic polynomial

In general, if α and β are the zeros of the quadratic polynomial $p(x) = ax^2 + bx + c$, $a \neq 0$, then by factor theorem we get, $x - \alpha$ and $x - \beta$ are the factors of $p(x)$.

Therefore, $\quad ax^2 + bx + c = k(x-\alpha)(x-\beta)$, where k is a non zero constant.

$$= k[x^2 - (\alpha+\beta)x + \alpha\beta]$$

Comparing the coefficients of x^2, x and the constant term on both sides, we obtain

$$a = k, \quad b = -k(\alpha + \beta) \quad \text{and} \quad c = k\alpha\beta$$

The basic relationships between the zeros and the coefficients of $p(x) = ax^2 + bx + c$ are

sum of zeros : $\alpha + \beta = -\dfrac{b}{a} = -\dfrac{\text{coefficient of } x}{\text{coefficient of } x^2}$.

product of zeros : $\alpha\beta = \dfrac{c}{a} = \dfrac{\text{constant term}}{\text{coefficient of } x^2}$.

Example 3.11

Find the zeros of the quadratic polynomial $x^2 + 9x + 20$, and verify the basic relationships between the zeros and the coefficients.

Solution Let $p(x) = x^2 + 9x + 20 = (x+4)(x+5)$

So, $p(x) = 0 \implies (x+4)(x+5) = 0 \qquad \therefore\ x = -4$ or $x = -5$

Thus, $p(-4) = (-4+4)(-4+5) = 0$ and $p(-5) = (-5+4)(-5+5) = 0$

Hence, the zeros of $p(x)$ are -4 and -5

Thus, sum of zeros $= -9$ and the product of zeros $= 20$ (1)

From the basic relationships, we get

the sum of the zeros $= -\dfrac{\text{coefficient of } x}{\text{coefficient of } x^2} = -\dfrac{9}{1} = -9$ (2)

product of the zeros $= \dfrac{\text{constant term}}{\text{coefficient of } x^2} = \dfrac{20}{1} = 20$ (3)

Thus, the basic relationships are verified.

Remarks

To factorize $x^2 + 9x + 20$, one can proceed as follows

$$20 \to 4,\ 5 \quad \because 4+5=9,\ 4\times5=20$$

$\dfrac{4}{1}$ $\dfrac{5}{1}$ constant term

coefficient of x^2

$(x+4)(x+5)$.

Note

A quadratic polynomial $p(x) = ax^2 + bx + c$ may have atmost two zeros.

Now, for any $a \neq 0$, $a(x^2 - (\alpha+\beta)x + \alpha\beta)$ is a polynomial with zeros α and β. Since we can choose any non zero a, there are infinitely many quadratic polynomials with zeros α and β.

Example 3.12

Find a quadratic polynomial if the sum and product of zeros of it are -4 and 3 respectively.

Solution Let α and β be the zeros of a quadratic polynomial.

Given that $\alpha + \beta = -4$ and $\alpha\beta = 3$.

One of the such polynomials is $p(x) = x^2 - (\alpha+\beta)x + \alpha\beta$

$$= x^2 - (-4)x + 3 = x^2 + 4x + 3$$

Example 3.13

Find a quadratic polynomial with zeros at $x = \frac{1}{4}$ and $x = -1$.

Solution

Let α and β be the zeros of $p(x)$
Using the relationship between zeros and coefficients, we have

$$p(x) = x^2 - (\alpha + \beta)x + \alpha\beta$$
$$= x^2 - \left(\frac{1}{4} - 1\right)x + \left(\frac{1}{4}\right)(-1)$$
$$= x^2 + \frac{3}{4}x - \frac{1}{4}$$

It is a polynomial with zeros $\frac{1}{4}$ and -1.

Aliter The required polynomial is obtained directly as follows:

$$p(x) = \left(x - \frac{1}{4}\right)(x + 1)$$
$$= x^2 + \frac{3}{4}x - \frac{1}{4}.$$

Any other polynomial with the desired property is obtained by multiplying $p(x)$ by any non-zero real number.

Note

$4x^2 + 3x - 1$ is also a polynomial with zeros $\frac{1}{4}$ and -1.

Exercise 3.3

1. Find the zeros of the following quadratic polynomials and verify the basic relationships between the zeros and the coefficients.

(i) $x^2 - 2x - 8$ (ii) $4x^2 - 4x + 1$ (iii) $6x^2 - 3 - 7x$ (iv) $4x^2 + 8x$

(v) $x^2 - 15$ (vi) $3x^2 - 5x + 2$ (vii) $2x^2 - 2\sqrt{2}x + 1$ (viii) $x^2 + 2x - 143$

2. Find a quadratic polynomial each with the given numbers as the sum and product of its zeros respectively.

(i) 3, 1 (ii) 2, 4 (iii) 0, 4 (iv) $\sqrt{2}, \frac{1}{5}$

(v) $\frac{1}{3}, 1$ (vi) $\frac{1}{2}, -4$ (vii) $\frac{1}{3}, -\frac{1}{3}$ (viii) $\sqrt{3}, 2$

3.4 Synthetic division

We know that when 29 is divided by 7 we get, 4 as the quotient and 1 as the remainder. Thus, 29 = 4(7) + 1. Similarly one can divide a polynomial $p(x)$ by another polynomial $q(x)$ which results in getting the quotient and remainder such that

$$p(x) = (\text{quotient})q(x) + \text{remainder}$$

That is, $p(x) = s(x)q(x) + r(x)$, where $\deg r(x) < \deg q(x)$.

This is called the Division Algorithm.

If $q(x) = x + a$, then $\deg r(x) = 0$. Thus, $r(x)$ is a constant.

Hence, $p(x) = s(x)(x + a) + r$, where r is a constant.

Now if we put $x = -a$ in the above, we have $p(-a) = s(-a)(-a + a) + r \implies r = p(-a)$.

Thus, if $q(x) = x + a$, then the remainder can be calculated by simply evaluating $p(x)$ at $x = -a$.

Division algorithm :

If $p(x)$ is the dividend and $q(x)$ is the divisor, then by division algorithm we write, $p(x) = s(x)q(x) + r(x)$.

Now, we have the following results.

(i) If $q(x)$ is linear , then $r(x)= r$ is a constant.

(ii) If $\deg q(x) = 1$ (i.e., $q(x)$ is linear), then $\deg p(x) = 1 + \deg s(x)$

(iii) If $p(x)$ is divided by $x + a$, then the remainder is $p(-a)$.

(iv) If $r = 0$, we say $q(x)$ divides $p(x)$ or equivalently $q(x)$ is a factor of $p(x)$.

Remarks

An elegant way of dividing a polynomial by a linear polynomial was introduced by Paolo Ruffin in 1809. His method is known as **synthetic division.** It facilitates the division of a polynomial by a linear polynomial with the help of the coefficients involved.

Paolo Ruffin
(1765-1822, Italy)

Let us explain the method of synthetic division with an example.

Let $p(x) = x^3 + 2x^2 - x - 4$ be the dividend and $q(x) = x + 2$ be the divisor. We shall find the quotient $s(x)$ and the remainder r, by proceeding as follows.

***Step* 1** Arrange the dividend and the divisor according to the descending powers of x and then write the coefficients of dividend in the first row (see figure). Insert 0 for missing terms.

$$x^3 + 2x^2 - x - 4$$

$$1 \quad 2 \quad -1 \quad -4$$

***Step* 2** Find out the zero of the divisor.

Step 3 Put 0 for the first entry in the 2^{nd} row.

Complete the entries of the 2^{nd} row and 3^{rd} row as shown below.

$$\begin{array}{r|cccc}
-2 & 1 & 2 & -1 & -4 \\
 & 0 & \xrightarrow{1\times(-2)} -2 & \xrightarrow{0\times(-2)} 0 & \xrightarrow{-1\times(-2)} 2 \\
\hline
 & 1+0 & 2+(-2) & -1+0 & -4+2 \\
 & =1 & =0 & =-1 & =-2 \longleftarrow \text{remainder}
\end{array}$$

Step 4 Write down the quotient and the remainder accordingly. All the entries except the last one in the third row constitute the coefficients of the quotient.

Thus, the quotient is $x^2 - 1$ and the remainder is –2.

Example 3.14

Find the quotient and remainder when $x^3 + x^2 - 7x - 3$ is divided by $x - 3$.

Solution Let $p(x) = x^3 + x^2 - 7x - 3$. The zero of the divisor is 3. So we consider,

$$\begin{array}{r|rrr|r} 3 & 1 & 1 & -7 & -3 \\ & 0 & 3 & 12 & 15 \\ \hline & 1 & 4 & 5 & 12 \end{array} \longrightarrow \text{Remainder.}$$

$\therefore$ When $p(x)$ is divided by $x - 3$, the quotient is $x^2 + 4x + 5$ and the remainder is 12.

Example 3.15

If the quotient on dividing $2x^4 + x^3 - 14x^2 - 19x + 6$ by $2x + 1$ is $x^3 + ax^2 - bx - 6$. Find the values of a and b, also the remainder.

Solution Let $p(x) = 2x^4 + x^3 - 14x^2 - 19x + 6$.

Given that the divisor is $2x + 1$. Write $2x + 1 = 0$. Then $x = -\frac{1}{2}$

$\therefore$ The zero of the divisor is $-\frac{1}{2}$.

$$\begin{array}{r|rrrr|r} -\frac{1}{2} & 2 & 1 & -14 & -19 & 6 \\ & 0 & -1 & 0 & 7 & 6 \\ \hline & 2 & 0 & -14 & -12 & 12 \end{array} \longrightarrow \text{Remainder}$$

So, $2x^4 + x^3 - 14x^2 - 19x + 6 = \left(x + \frac{1}{2}\right)\{2x^3 - 14x - 12\} + 12$

$= (2x + 1)\frac{1}{2}(2x^3 - 14x - 12) + 12$

Thus, the quotient is $\frac{1}{2}(2x^3 - 14x - 12) = x^3 - 7x - 6$ and the remainder is 12.

But, given quotient is $x^3 + ax^2 - bx - 6$. Comparing this with the quotient obtained we get, $a = 0$ and $b = 7$. Thus, $a = 0$, $b = 7$ and the remainder is 12.

Exercise 3.4

1. Find the quotient and remainder using synthetic division.
   (i) $(x^3 + x^2 - 3x + 5) \div (x - 1)$ (ii) $(3x^3 - 2x^2 + 7x - 5) \div (x + 3)$
   (iii) $(3x^3 + 4x^2 - 10x + 6) \div (3x - 2)$ (iv) $(3x^3 - 4x^2 - 5) \div (3x + 1)$
   (v) $(8x^4 - 2x^2 + 6x - 5) \div (4x + 1)$ (vi) $(2x^4 - 7x^3 - 13x^2 + 63x - 48) \div (2x - 1)$
2. If the quotient on dividing $x^4 + 10x^3 + 35x^2 + 50x + 29$ by $x + 4$ is $x^3 - ax^2 + bx + 6$, then find a, b and also the remainder.
3. If the quotient on dividing, $8x^4 - 2x^2 + 6x - 7$ by $2x + 1$ is $4x^3 + px^2 - qx + 3$, then find p, q and also the remainder.

3.4.1 Factorization using synthetic division

We have already learnt in class IX, how to factorize quadratic polynomials. In this section, let us learn, how to factorize the cubic polynomial using synthetic division.

If we identify one linear factor of cubic polynomial $p(x)$, then using synthetic division we get the quadratic factor of $p(x)$. Further if possible one can factorize the quadratic factor into two linear factors. Hence the method of synthetic division helps us to factorize a cubic polynomial into linear factors if it can be factorized.

Note

(i) For any polynomial $p(x)$, $x = a$ is zero if and only if $p(a) = 0$.

(ii) $x - a$ is a factor for $p(x)$ if and only if $p(a) = 0$. (Factor theorem)

(iii) $x - 1$ is a factor of $p(x)$ if and only if the sum of coefficients of $p(x)$ is 0.

(iv) $x + 1$ is a factor of $p(x)$ if and only if sum of the coefficients of even powers of x, including constant is equal to sum of the coefficients of odd powers of x.

Example 3.16

(i) Prove that $x - 1$ is a factor of $x^3 - 6x^2 + 11x - 6$.

(ii) Prove that $x + 1$ is a factor of $x^3 + 6x^2 + 11x + 6$.

Solution

(i) Let $p(x) = x^3 - 6x^2 + 11x - 6$.

$p(1) = 1 - 6 + 11 - 6 = 0$. (note that sum of the coefficients is 0)

Thus, $(x - 1)$ is a factor of $p(x)$.

(ii) Let $q(x) = x^3 + 6x^2 + 11x + 6$.

$q(-1) = -1 + 6 - 11 + 6 = 0$. Hence, $x + 1$ is a factor of $q(x)$

Example 3.17

Factorize $2x^3 - 3x^2 - 3x + 2$ into linear factors.

Solution Let $p(x) = 2x^3 - 3x^2 - 3x + 2$

Now, $p(1) = -2 \neq 0$ (note that sum of the coefficients is not zero)

$\therefore$ $(x - 1)$ is not a factor of $p(x)$.

However, $p(-1) = 2(-1)^3 - 3(-1)^2 - 3(-1) + 2 = 0$.

So, $x + 1$ is a factor of $p(x)$.

We shall use synthetic division to find the other factors.

$$\begin{array}{r|rrrr} -1 & 2 & -3 & -3 & 2 \\ & 0 & -2 & 5 & -2 \\ \hline & 2 & -5 & 2 & \boxed{0} \end{array} \longrightarrow \text{Remainder}$$

Thus, $p(x) = (x + 1)(2x^2 - 5x + 2)$

Now, $2x^2 - 5x + 2 = 2x^2 - 4x - x + 2 = (x - 2)(2x - 1)$.

Hence, $2x^3 - 3x^2 - 3x + 2 = (x + 1)(x - 2)(2x - 1)$.

Remarks

To factorize $2x^2 - 5x + 2$, one can proceed as follows

$$\begin{array}{ccc} & 4 & \\ -4 & & -1 \end{array} \quad \because -4 + (-1) = -5,\ -4 \times (-1) = 4$$

$$\frac{-4}{2} = \frac{-2}{1} \qquad \frac{-1}{2}$$

constant term

coefficient of x^2

$(x - 2)(2x - 1)$.

Example 3.18

Factorize $x^3 - 3x^2 - 10x + 24$

Solution Let $p(x) = x^3 - 3x^2 - 10x + 24$.

Since $p(1) \neq 0$ and $p(-1) \neq 0$, neither $x+1$ nor $x-1$ is a factor of $p(x)$.

Therefore, we have to search for different values of x by trial and error method.

When $x = 2$, $p(2) = 0$. Thus, $x - 2$ is a factor of $p(x)$.

To find the other factors, let us use the synthetic division.

$$\begin{array}{r|rrrr} 2 & 1 & -3 & -10 & 24 \\ & 0 & 2 & -2 & -24 \\ \hline & 1 & -1 & -12 & \boxed{0} \end{array} \longrightarrow \text{Remainder.}$$

$\therefore$ The other factor is $x^2 - x - 12$.

Now, $x^2 - x - 12 = x^2 - 4x + 3x - 12 = (x-4)(x+3)$

Hence, $x^3 - 3x^2 - 10x + 24 = (x-2)(x+3)(x-4)$

Exercise 3.5

1. Factorize each of the following polynomials.

(i) $x^3 - 2x^2 - 5x + 6$	(ii) $4x^3 - 7x + 3$	(iii) $x^3 - 23x^2 + 142x - 120$
(iv) $4x^3 - 5x^2 + 7x - 6$	(v) $x^3 - 7x + 6$	(vi) $x^3 + 13x^2 + 32x + 20$
(vii) $2x^3 - 9x^2 + 7x + 6$	(viii) $x^3 - 5x + 4$	(ix) $x^3 - 10x^2 - x + 10$
(x) $2x^3 + 11x^2 - 7x - 6$	(xi) $x^3 + x^2 + x - 14$	(xii) $x^3 - 5x^2 - 2x + 24$

3.5 Greatest Common Divisor (GCD) and Least Common Multiple (LCM)

3.5.1 Greatest Common Divisor (GCD)

The Highest Common Factor (HCF) or Greatest Common Divisor (GCD) of two or more algebraic expressions is the expression of highest degree which divides each of them without remainder.

Consider the simple expressions

(i) a^4, a^3, a^5, a^6 (ii) a^3b^4, ab^5c^2, a^2b^7c

In (i), note that a, a^2, a^3 are the divisors of all these expressions. Out of them, a^3 is the divisor with highest power. Therefore a^3 is the GCD of the expressions a^4, a^3, a^5, a^6.

In (ii), similarly, one can easily see that ab^4 is the GCD of a^3b^4, ab^5c^2, a^2b^7c.

If the expressions have numerical coefficients, find their greatest common divisor, and prefix it as a coefficient to the greatest common divisor of the algebraic expressions.

Let us consider a few more examples to understand the greatest common divisor.

Examples 3.19

Find the GCD of the following : (i) 90, 150, 225 (ii) $15x^4y^3z^5$, $12x^2y^7z^2$

(iii) $6(2x^2 - 3x - 2)$, $8(4x^2 + 4x + 1)$, $12(2x^2 + 7x + 3)$

Solution

(i) Let us write the numbers 90, 150 and 225 in the product of their prime factors as

$$90 = 2 \times 3 \times 3 \times 5\ ,\ 150 = 2 \times 3 \times 5 \times 5 \text{ and } 225 = 3 \times 3 \times 5 \times 5$$

From the above 3 and 5 are common prime factors of all the given numbers.

Hence the GCD $= 3 \times 5 = 15$

(ii) We shall use similar technique to find the GCD of algebraic expressions.

Now let us take the given expressions $15x^4y^3z^5$ and $12x^2y^7z^2$.

Here the common divisors of the given expressions are 3, x^2, y^3 and z^2.

Therefore, GCD $= 3 \times x^2 \times y^3 \times z^2 = 3x^2y^3z^2$

(iii) Given expressions are $6(2x^2 - 3x - 2)$, $8(4x^2 + 4x + 1)$, $12(2x^2 + 7x + 3)$

Now, GCD of 6, 8, 12 is 2

Next let us find the factors of quadratic expressions.

$$2x^2 - 3x - 2 = (2x + 1)(x - 2)$$

$$4x^2 + 4x + 1 = (2x + 1)(2x + 1)$$

$$2x^2 + 7x + 3 = (2x + 1)(x + 3)$$

Common factor of the above quadratic expressions is $(2x + 1)$.

Therefore, GCD $= 2(2x + 1)$.

3.5.2 Greatest common divisor of polynomials using division algorithm

First let us consider the simple case of finding GCD of 924 and 105.

$924 = 8 \times 105 + 84$

$105 = 1 \times 84 + 21,$ (or)

$84 = 4 \times 21 + 0,$

21 is the GCD of 924 and 105

$$105\overline{\big)\,924\,}^{\,8} \quad \begin{array}{r} 924 \\ 840 \\ \hline 84 \end{array} \qquad 84\overline{\big)\,105\,}^{\,1} \quad \begin{array}{r} 105 \\ 84 \\ \hline 21 \end{array} \qquad 21\overline{\big)\,84\,}^{\,4} \quad \begin{array}{r} 84 \\ 84 \\ \hline 0 \end{array}$$

Similar technique works with polynomials when they have GCD.

Let $f(x)$ and $g(x)$ be two non constant polynomials with $\deg(f(x)) \geq \deg(g(x))$. We want to find GCD of $f(x)$ and $g(x)$. If $f(x)$ and $g(x)$ can be factored into linear irreducible quadratic polynomials, then we can easily find the GCD by the method which we have learnt above. If the polynomials $f(x)$ and $g(x)$ are not easily factorable, then it will be a difficult problem.

However, the following method gives a systematic way of finding GCD.

Step 1 First, divide $f(x)$ by $g(x)$ to obtain $f(x) = g(x)q(x) + r(x)$ where $q(x)$ is the quotient and $r(x)$ is remainder, so $\deg(g(x)) > \deg(r(x))$

If the remainder $r(x)$ is 0, then $g(x)$ is the GCD of $f(x)$ and $g(x)$.

Step 2 If the remainder $r(x)$ is non-zero, divide $g(x)$ by $r(x)$ to obtain $g(x) = r(x)q(x) + r_1(x)$ where $r_1(x)$ is the remainder. So $\deg r(x) > \deg r_1(x)$.

If the remainder $r_1(x)$ is 0, then $r(x)$ is the required GCD.

Step 3 If $r_1(x)$ is non-zero, then continue the process until we get zero as remainder. The remainder in the last but one step is the GCD of $f(x)$ and $g(x)$.

We write GCD($f(x)$, $g(x)$) to denote the GCD of the polynomials $f(x)$ and $g(x)$

Remarks

Euclid's division algorithm is based on the principle that GCD of two numbers does not change if the small number is subtracted from the larger number. Thus, GCD (252,105) = GCD (147,105) = GCD(42,105) = GCD(63,42) = GCD(21,42) = 21.

Example 3.20

Find the GCD of the polynomials $x^4 + 3x^3 - x - 3$ and $x^3 + x^2 - 5x + 3$.

Solution Let $f(x) = x^4 + 3x^3 - x - 3$ and $g(x) = x^3 + x^2 - 5x + 3$

Here degree of $f(x) >$ degree of $g(x)$. $\therefore$ Divisor is $x^3 + x^2 - 5x + 3$

$$\begin{array}{r|l}
 & x + 2 \\ \hline
x^3 + x^2 - 5x + 3 & x^4 + 3x^3 + 0x^2 - x - 3 \\
 & x^4 + x^3 - 5x^2 + 3x \\ \hline
 & 2x^3 + 5x^2 - 4x - 3 \\
 & 2x^3 + 2x^2 - 10x + 6 \\ \hline
 & 3x^2 + 6x - 9 \\
 & \Longrightarrow \quad x^2 + 2x - 3 \longrightarrow \text{remainder}(\neq 0)
\end{array}
\qquad
\begin{array}{r|l}
 & x - 1 \\ \hline
x^2 + 2x - 3 & x^3 + x^2 - 5x + 3 \\
 & x^3 + 2x^2 - 3x \\ \hline
 & -x^2 - 2x + 3 \\
 & -x^2 - 2x + 3 \\ \hline
 & 0 \longrightarrow \text{remainder}
\end{array}$$

Therefore, GCD ($f(x)$, $g(x)$) $= x^2 + 2x - 3$.

Remarks

The two original expressions have no simple factors (constants). Thus their GCD can have none. Hence, in the above example we removed the simple factor 3 from $3x^2 + 6x - 9$ and took $x^2 + 2x - 3$ as the new divisor.

Example 3.21

Find the GCD of the following polynomials

$3x^4 + 6x^3 - 12x^2 - 24x$ and $4x^4 + 14x^3 + 8x^2 - 8x$.

Solution Let $f(x) = 3x^4 + 6x^3 - 12x^2 - 24x = 3x(x^3 + 2x^2 - 4x - 8)$.

Let $g(x) = 4x^4 + 14x^3 + 8x^2 - 8x = 2x(2x^3 + 7x^2 + 4x - 4)$

Let us find the GCD for the polynomials $x^3 + 2x^2 - 4x - 8$ and $2x^3 + 7x^2 + 4x - 4$

We choose the divisor to be $x^3 + 2x^2 - 4x - 8$.

$$\begin{array}{r|l} & \quad 2 \\ \hline x^3 + 2x^2 - 4x - 8 & 2x^3 + 7x^2 + 4x - 4 \\ & 2x^3 + 4x^2 - 8x - 16 \\ \hline & \quad 3x^2 + 12x + 12 \\ & \quad (x^2 + 4x + 4) \\ & \qquad \rightarrow \text{remainder} (\neq 0) \end{array} \qquad \begin{array}{r|l} & \quad x - 2 \\ \hline x^2 + 4x + 4 & x^3 + 2x^2 - 4x - 8 \\ & x^3 + 4x^2 + 4x \\ \hline & \quad -2x^2 - 8x - 8 \\ & \quad -2x^2 - 8x - 8 \\ \hline & \qquad 0 \longrightarrow \text{remainder} \end{array}$$

Common factor of $x^3 + 2x^2 - 4x - 8$ and $2x^3 + 7x^2 + 4x - 4$ is $x^2 + 4x + 4$

Also common factor of $3x$ and $2x$ is x.

Thus, GCD $(f(x), g(x)) = x(x^2 + 4x + 4)$.

Exercise 3.6

1. Find the greatest common divisor of
   (i) $7x^2yz^4$, $21x^2y^5z^3$ (ii) x^2y, x^3y, x^2y^2
   (iii) $25bc^4d^3$, $35b^2c^5$, $45c^3d$ (iv) $35x^5y^3z^4$, $49x^2yz^3$, $14xy^2z^2$

2. Find the GCD of the following
   (i) $c^2 - d^2$, $c(c - d)$ (ii) $x^4 - 27a^3x$, $(x - 3a)^2$
   (iii) $m^2 - 3m - 18$, $m^2 + 5m + 6$ (iv) $x^2 + 14x + 33$, $x^3 + 10x^2 - 11x$
   (v) $x^2 + 3xy + 2y^2$, $x^2 + 5xy + 6y^2$ (vi) $2x^2 - x - 1$, $4x^2 + 8x + 3$
   (vii) $x^2 - x - 2$, $x^2 + x - 6$, $3x^2 - 13x + 14$ (viii) $x^3 - x^2 + x - 1$, $x^4 - 1$
   (ix) $24(6x^4 - x^3 - 2x^2)$, $20(2x^6 + 3x^5 + x^4)$
   (x) $(a - 1)^5(a + 3)^2$, $(a - 2)^2(a - 1)^3(a + 3)^4$

3. Find the GCD of the following pairs of polynomials using division algorithm.
   (i) $x^3 - 9x^2 + 23x - 15$, $4x^2 - 16x + 12$
   (ii) $3x^3 + 18x^2 + 33x + 18$, $3x^2 + 13x + 10$
   (iii) $2x^3 + 2x^2 + 2x + 2$, $6x^3 + 12x^2 + 6x + 12$
   (iv) $x^3 - 3x^2 + 4x - 12$, $x^4 + x^3 + 4x^2 + 4x$

3.5.3 Least Common Multiple (LCM)

The least common multiple of two or more algebraic expressions is the expression of lowest degree which is divisible by each of them without remainder. For example, consider the simple expressions a^4, a^3, a^6.

Now, $a^6,\ a^7,\ a^8, \cdots$ are common multiples of a^3, a^4 and a^6.

Of all the common multiples, the least common multiple is a^6

Hence LCM of a^4, a^3, a^6 is a^6. Similarly, a^3b^7 is the LCM of $a^3b^4,\ ab^5,\ a^2b^7$.

We shall consider some more examples of finding LCM.

Example 3.22

Find the LCM of the following.

(i) 90, 150, 225 (ii) $35a^2c^3b,\ 42a^3cb^2,\ 30ac^2b^3$

(iii) $(a-1)^5(a+3)^2,\ (a-2)^2(a-1)^3(a+3)^4$

(iv) $x^3+y^3,\ x^3-y^3,\ x^4+x^2y^2+y^4$

Solution

(i) Now,

$$90 = 2\times3\times3\times5 = 2^1\times3^2\times5^1$$
$$150 = 2\times3\times5\times5 = 2^1\times3^1\times5^2$$
$$225 = 3\times3\times5\times5 = 3^2\times5^2$$

The product $2^1\times3^2\times5^2 = 450$ is the required LCM.

(ii) Now, LCM of 35, 42 and 30 is $5\times7\times6 = 210$

Hence, the required LCM $= 210\times a^3\times c^3\times b^3 = 210a^3c^3b^3$.

(iii) Now, LCM of $(a-1)^5(a+3)^2,\ (a-2)^2(a-1)^3(a+3)^4$ is $(a-1)^5(a+3)^4(a-2)^2$.

(iv) Let us first find the factors for each of the given expressions.

$$x^3+y^3 = (x+y)(x^2-xy+y^2)$$
$$x^3-y^3 = (x-y)(x^2+xy+y^2)$$
$$x^4+x^2y^2+y^4 = (x^2+y^2)^2-x^2y^2 = (x^2+xy+y^2)(x^2-xy+y^2)$$

Thus,
$$\text{LCM} = (x+y)(x^2-xy+y^2)(x-y)(x^2+xy+y^2)$$
$$= (x^3+y^3)(x^3-y^3) = x^6-y^6.$$

Exercise 3.7

Find the LCM of the following.

1. $x^3y^2,\ xyz$
2. $3x^2yz,\ 4x^3y^3$
3. $a^2bc,\ b^2ca,\ c^2ab$
4. $66a^4b^2c^3,\ 44a^3b^4c^2,\ 24a^2b^3c^4$
5. $a^{m+1},\ a^{m+2},\ a^{m+3}$
6. $x^2y+xy^2,\ x^2+xy$
7. $3(a-1),\ 2(a-1)^2,\ (a^2-1)$
8. $2x^2-18y^2,\ 5x^2y+15xy^2,\ x^3+27y^3$
9. $(x+4)^2(x-3)^3,\ (x-1)(x+4)(x-3)^2$
10. $10(9x^2+6xy+y^2),\ 12(3x^2-5xy-2y^2),\ 14(6x^4+2x^3)$.

3.5.4 Relation between LCM and GCD

We know that the product of two positive integers is equal to the product of their LCM and GCD. For example, $21 \times 35 = 105 \times 7$, where LCM (21,35) =105 and GCD (21,35) = 7.

In the same way, we have the following result:

The product of any two polynomials is equal to the product of their LCM and GCD.

That is, $f(x) \times g(x) = \text{LCM}\,(f(x), g(x)) \times \text{GCD}\,(f(x), g(x))$.

Let us justify this result with an example.

Let $f(x) = 12(x^4 - x^3)$ and $g(x) = 8(x^4 - 3x^3 + 2x^2)$ be two polynomials.

Now, $f(x) = 12(x^4 - x^3) = 2^2 \times 3 \times x^3 \times (x-1)$ (1)

Also, $g(x) = 8(x^4 - 3x^3 + 2x^2) = 2^3 \times x^2 \times (x-1) \times (x-2)$ (2)

From (1) and (2) we get,

$$\text{LCM}\,(f(x), g(x)) = 2^3 \times 3^1 \times x^3 \times (x-1) \times (x-2) = 24x^3(x-1)(x-2)$$

$$\text{GCD}\,(f(x), g(x)) = 4x^2(x-1)$$

Therefore, $\text{LCM} \times \text{GCD} = 24x^3(x-1)(x-2) \times 4x^2(x-1)$

$= 96x^5(x-1)^2(x-2)$ (3)

Also, $f(x) \times g(x) = 12x^3(x-1) \times 8x^2(x-1)(x-2)$

$= 96x^5(x-1)^2(x-2)$ (4)

From (3) and (4) we obtain, $\text{LCM} \times \text{GCD} = f(x) \times g(x)$.

Thus, the product of LCM and GCD of two polynomials is equal to the product of the two polynomials. Further, if $f(x)$, $g(x)$ and one of LCM and GCD are given, then the other can be found without ambiguity because LCM and GCD are unique, except for a factor of –1.

Example 3.23

The GCD of $x^4 + 3x^3 + 5x^2 + 26x + 56$ and $x^4 + 2x^3 - 4x^2 - x + 28$ is $x^2 + 5x + 7$. Find their LCM.

Solution Let $f(x) = x^4 + 3x^3 + 5x^2 + 26x + 56$ and $g(x) = x^4 + 2x^3 - 4x^2 - x + 28$

Given that $\text{GCD} = x^2 + 5x + 7$. Also, we have $\text{GCD} \times \text{LCM} = f(x) \times g(x)$.

Thus, $\text{LCM} = \dfrac{f(x) \times g(x)}{\text{GCD}}$ (1)

Now, GCD divides both $f(x)$ and $g(x)$.

Let us divide $f(x)$ by the GCD.

	1	–2	8		
1 5 7	1	3	5	26	56
	1	5	7		
		–2	–2	26	
		–2	–10	–14	
			8	40	56
			8	40	56
				0	

When $f(x)$ is divided by GCD, we get the quotient as $x^2 - 2x + 8$.

Now, (1) $\Longrightarrow$ LCM $= (x^2 - 2x + 8) \times g(x)$

Thus, LCM $= (x^2 - 2x + 8)(x^4 + 2x^3 - 4x^2 - x + 28)$.

Note

In the above problem, we can also divide $g(x)$ by GCD and multiply the quotient by $f(x)$ to get the required LCM.

Example 3.24

The GCD and LCM of two polynomials are $x + 1$ and $x^6 - 1$ respectively. If one of the polynomials is $x^3 + 1$, find the other.

Solution Given GCD = $x + 1$ and LCM = $x^6 - 1$

Let $f(x) = x^3 + 1$.

We know that LCM $\times$ GCD $= f(x) \times g(x)$

$$\Longrightarrow g(x) = \frac{\text{LCM} \times \text{GCD}}{f(x)} = \frac{(x^6 - 1)(x + 1)}{x^3 + 1}$$

$$= \frac{(x^3 + 1)(x^3 - 1)(x + 1)}{x^3 + 1} = (x^3 - 1)(x + 1)$$

Hence, $g(x) = (x^3 - 1)(x + 1)$.

Exercise 3.8

1. Find the LCM of each pair of the following polynomials.
   (i) $x^2 - 5x + 6,\ x^2 + 4x - 12$ whose GCD is $x - 2$.
   (ii) $x^4 + 3x^3 + 6x^2 + 5x + 3,\ x^4 + 2x^2 + x + 2$ whose GCD is $x^2 + x + 1$.
   (iii) $2x^3 + 15x^2 + 2x - 35,\ x^3 + 8x^2 + 4x - 21$ whose GCD is $x + 7$.
   (iv) $2x^3 - 3x^2 - 9x + 5,\ 2x^4 - x^3 - 10x^2 - 11x + 8$ whose GCD is $2x - 1$.

2. Find the other polynomial $q(x)$ of each of the following, given that LCM and GCD and one polynomial $p(x)$ respectively.
   (i) $(x + 1)^2(x + 2)^2,\ (x + 1)(x + 2),\ (x + 1)^2(x + 2)$.
   (ii) $(4x + 5)^3(3x - 7)^3,\ (4x + 5)(3x - 7)^2,\ (4x + 5)^3(3x - 7)^2$.
   (iii) $(x^4 - y^4)(x^4 + x^2y^2 + y^4),\ x^2 - y^2,\ x^4 - y^4$.
   (iv) $(x^3 - 4x)(5x + 1),\ (5x^2 + x),\ (5x^3 - 9x^2 - 2x)$.
   (v) $(x - 1)(x - 2)(x^2 - 3x + 3),\ (x - 1),\ (x^3 - 4x^2 + 6x - 3)$.
   (vi) $2(x + 1)(x^2 - 4),\ (x + 1),\ (x + 1)(x - 2)$.

3.6 Rational expressions

A rational number is defined as a quotient $\frac{m}{n}$, of two integers m and $n \neq 0$. Similarly a rational expression is a quotient $\frac{p(x)}{q(x)}$ of two polynomials $p(x)$ and $q(x)$, where $q(x)$ is a non zero polynomial.

Every polynomial $p(x)$ is a rational expression, since $p(x)$ can be written as $\frac{p(x)}{1}$ where 1 is the constant polynomial.

However, a rational expression need not be a polynomial, for example $\frac{x}{x^2+1}$ is a rational expression but not a polynomial. Some examples of rational expressions are $2x+7$, $\frac{3x+2}{x^2+x+1}$, $\frac{x^3+\sqrt{2}x+5}{x^2+x-\sqrt{3}}$.

3.6.1 Rational expressions in lowest form

If the two polynomials $p(x)$ and $q(x)$ have the integer coefficients such that GCD of $p(x)$ and $q(x)$ is 1, then we say that $\frac{p(x)}{q(x)}$ is a rational expression in its lowest terms.

If a rational expression is not in its lowest terms, then it can be reduced to its lowest terms by dividing both numerator $p(x)$ and denominator $q(x)$ by the GCD of $p(x)$ and $q(x)$. Let us consider some examples.

Example 3.25

Simplify the rational expressions into lowest forms.

(i) $\frac{5x+20}{7x+28}$ (ii) $\frac{x^3-5x^2}{3x^3+2x^4}$

(iii) $\frac{6x^2-5x+1}{9x^2+12x-5}$ (iv) $\frac{(x-3)(x^2-5x+4)}{(x-1)(x^2-2x-3)}$

Solution

(i) Now, $\frac{5x+20}{7x+28} = \frac{5(x+4)}{7(x+4)} = \frac{5}{7}$

(ii) Now, $\frac{x^3-5x^2}{3x^3+2x^4} = \frac{x^2(x-5)}{x^3(2x+3)} = \frac{x-5}{x(2x+3)}$

(iii) Let $p(x) = 6x^2-5x+1 = (2x-1)(3x-1)$ and

$q(x) = 9x^2+12x-5 = (3x+5)(3x-1)$

Therefore, $\frac{p(x)}{q(x)} = \frac{(2x-1)(3x-1)}{(3x+5)(3x-1)} = \frac{2x-1}{3x+5}$

(iv) Let $f(x) = (x-3)(x^2-5x+4) = (x-3)(x-1)(x-4)$ and

$g(x) = (x-1)(x^2-2x-3) = (x-1)(x-3)(x+1)$

Therefore, $\frac{f(x)}{g(x)} = \frac{(x-3)(x-1)(x-4)}{(x-1)(x-3)(x+1)} = \frac{x-4}{x+1}$

Exercise 3.9

Simplify the following into their lowest forms.

(i) $\dfrac{6x^2+9x}{3x^2-12x}$ (ii) $\dfrac{x^2+1}{x^4-1}$ (iii) $\dfrac{x^3-1}{x^2+x+1}$

(iv) $\dfrac{x^3-27}{x^2-9}$ (v) $\dfrac{x^4+x^2+1}{x^2+x+1}$ (Hint: $x^4+x^2+1=(x^2+1)^2-x^2$)

(vi) $\dfrac{x^3+8}{x^4+4x^2+16}$ (vii) $\dfrac{2x^2+x-3}{2x^2+5x+3}$ (viii) $\dfrac{2x^4-162}{(x^2+9)(2x-6)}$

(ix) $\dfrac{(x-3)(x^2-5x+4)}{(x-4)(x^2-2x-3)}$ (x) $\dfrac{(x-8)(x^2+5x-50)}{(x+10)(x^2-13x+40)}$ (xi) $\dfrac{4x^2+9x+5}{8x^2+6x-5}$

(xii) $\dfrac{(x-1)(x-2)(x^2-9x+14)}{(x-7)(x^2-3x+2)}$

3.6.2 Multiplication and division of rational expressions

If $\dfrac{p(x)}{q(x)}$; $q(x)\neq 0$ and $\dfrac{g(x)}{h(x)}$; $h(x)\neq 0$ are two rational expressions, then

(i) their product $\dfrac{p(x)}{q(x)}\times\dfrac{g(x)}{h(x)}$ is defined as $\dfrac{p(x)\times g(x)}{q(x)\times h(x)}$

(ii) their division $\dfrac{p(x)}{q(x)}\div\dfrac{g(x)}{h(x)}$ is defined as $\dfrac{p(x)}{q(x)}\times\dfrac{h(x)}{g(x)}$.

Thus, $\dfrac{p(x)}{q(x)}\div\dfrac{g(x)}{h(x)} = \dfrac{p(x)\times h(x)}{q(x)\times g(x)}$

Example 3.26

Multiply (i) $\dfrac{x^3y^2}{9z^4}$ by $\dfrac{27z^5}{x^4y^2}$ (ii) $\dfrac{a^3+b^3}{a^2+2ab+b^2}$ by $\dfrac{a^2-b^2}{a-b}$ (iii) $\dfrac{x^3-8}{x^2-4}$ by $\dfrac{x^2+6x+8}{x^2+2x+4}$

Solution

(i) Now, $\dfrac{x^3y^2}{9z^4}\times\dfrac{27z^5}{x^4y^2} = \dfrac{(x^3y^2)(27z^5)}{(9z^4)(x^4y^2)} = \dfrac{3z}{x}$.

(ii) $\dfrac{a^3+b^3}{a^2+2ab+b^2}\times\dfrac{a^2-b^2}{a-b} = \dfrac{(a+b)(a^2-ab+b^2)}{(a+b)(a+b)}\times\dfrac{(a+b)(a-b)}{(a-b)} = a^2-ab+b^2$.

(iii) Now, $\dfrac{x^3-8}{x^2-4}\times\dfrac{x^2+6x+8}{x^2+2x+4} = \dfrac{x^3-2^3}{x^2-2^2}\times\dfrac{(x+4)(x+2)}{x^2+2x+4}$

$$= \frac{(x-2)(x^2+2x+4)}{(x+2)(x-2)}\times\frac{(x+4)(x+2)}{x^2+2x+4} = x+4.$$

Example 3.27

Divide (i) $\dfrac{4x-4}{x^2-1}$ by $\dfrac{x-1}{x+1}$ (ii) $\dfrac{x^3-1}{x+3}$ by $\dfrac{x^2+x+1}{3x+9}$ (iii) $\dfrac{x^2-1}{x^2-25}$ by $\dfrac{x^2-4x-5}{x^2+4x-5}$

Solution

(i) $\dfrac{4x-4}{x^2-1} \div \dfrac{x-1}{x+1} = \dfrac{4(x-1)}{(x+1)(x-1)} \times \dfrac{(x+1)}{(x-1)} = \dfrac{4}{x-1}.$

(ii) $\dfrac{x^3-1}{x+3} \div \dfrac{x^2+x+1}{3x+9} = \dfrac{(x-1)(x^2+x+1)}{x+3} \times \dfrac{3(x+3)}{x^2+x+1} = 3(x-1).$

(iii) $\dfrac{x^2-1}{x^2-25} \div \dfrac{x^2-4x-5}{x^2+4x-5} = \dfrac{(x+1)(x-1)}{(x+5)(x-5)} \times \dfrac{(x+5)(x-1)}{(x-5)(x+1)}$

$= \dfrac{(x-1)(x-1)}{(x-5)(x-5)} = \dfrac{x^2-2x+1}{x^2-10x+25}.$

Exercise 3.10

1. Multiply the following and write your answer in lowest terms.

(i) $\dfrac{x^2-2x}{x+2} \times \dfrac{3x+6}{x-2}$ (ii) $\dfrac{x^2-81}{x^2-4} \times \dfrac{x^2+6x+8}{x^2-5x-36}$

(iii) $\dfrac{x^2-3x-10}{x^2-x-20} \times \dfrac{x^2-2x+4}{x^3+8}$ (iv) $\dfrac{x^2-16}{x^2-3x+2} \times \dfrac{x^2-4}{x^3+64} \times \dfrac{x^2-4x+16}{x^2-2x-8}$

(v) $\dfrac{3x^2+2x-1}{x^2-x-2} \times \dfrac{2x^2-3x-2}{3x^2+5x-2}$ (vi) $\dfrac{2x-1}{x^2+2x+4} \times \dfrac{x^4-8x}{2x^2+5x-3} \times \dfrac{x+3}{x^2-2x}$

2. Divide the following and write your answer in lowest terms.

(i) $\dfrac{x}{x+1} \div \dfrac{x^2}{x^2-1}$ (ii) $\dfrac{x^2-36}{x^2-49} \div \dfrac{x+6}{x+7}$

(iii) $\dfrac{x^2-4x-5}{x^2-25} \div \dfrac{x^2-3x-10}{x^2+7x+10}$ (iv) $\dfrac{x^2+11x+28}{x^2-4x-77} \div \dfrac{x^2+7x+12}{x^2-2x-15}$

(v) $\dfrac{2x^2+13x+15}{x^2+3x-10} \div \dfrac{2x^2-x-6}{x^2-4x+4}$ (vi) $\dfrac{3x^2-x-4}{9x^2-16} \div \dfrac{4x^2-4}{3x^2-2x-1}$

(vii) $\dfrac{2x^2+5x-3}{2x^2+9x+9} \div \dfrac{2x^2+x-1}{2x^2+x-3}$

3.6.3 Addition and subtraction of rational expressions

If $\dfrac{p(x)}{q(x)}$ and $\dfrac{r(x)}{s(x)}$ are any two rational expressions with $q(x) \neq 0$ and $s(x) \neq 0$, then we define the sum and the difference (subtraction) as

$$\frac{p(x)}{q(x)} \pm \frac{r(x)}{s(x)} = \frac{p(x).s(x) \pm q(x)r(x)}{q(x).s(x)}$$

Example 3.28

Simplify (i) $\dfrac{x+2}{x+3} + \dfrac{x-1}{x-2}$ (ii) $\dfrac{x+1}{(x-1)^2} + \dfrac{1}{x+1}$ (iii) $\dfrac{x^2-x-6}{x^2-9} + \dfrac{x^2+2x-24}{x^2-x-12}$

Solution

(i) $\dfrac{x+2}{x+3}+\dfrac{x-1}{x-2} = \dfrac{(x+2)(x-2)+(x-1)(x+3)}{(x+3)(x-2)} = \dfrac{2x^2+2x-7}{x^2+x-6}$

(ii) $\dfrac{x+1}{(x-1)^2}+\dfrac{1}{x+1} = \dfrac{(x+1)^2+(x-1)^2}{(x-1)^2(x+1)} = \dfrac{2x^2+2}{(x-1)^2(x+1)}$

$$= \dfrac{2x^2+2}{x^3-x^2-x+1}$$

(iii) $\dfrac{x^2-x-6}{x^2-9}+\dfrac{x^2+2x-24}{x^2-x-12} = \dfrac{(x-3)(x+2)}{(x+3)(x-3)}+\dfrac{(x+6)(x-4)}{(x+3)(x-4)}$

$$= \dfrac{x+2}{x+3}+\dfrac{x+6}{x+3} = \dfrac{x+2+x+6}{x+3} = \dfrac{2x+8}{x+3}$$

Example 3.29

What rational expression should be added to $\dfrac{x^3-1}{x^2+2}$ to get $\dfrac{2x^3-x^2+3}{x^2+2}$?

Solution Let $p(x)$ be the required rational expression.

Given that $\dfrac{x^3-1}{x^2+2} + p(x) = \dfrac{2x^3-x^2+3}{x^2+2}$

$$p(x) = \dfrac{2x^3-x^2+3}{x^2+2} - \dfrac{x^3-1}{x^2+2}$$

$$= \dfrac{2x^3-x^2+3-x^3+1}{x^2+2} = \dfrac{x^3-x^2+4}{x^2+2}$$

Example 3.30

Simplify $\left(\dfrac{2x-1}{x-1}-\dfrac{x+1}{2x+1}\right)+\dfrac{x+2}{x+1}$ as a quotient of two polynomials in the simplest form.

Solution Now, $\left(\dfrac{2x-1}{x-1}-\dfrac{x+1}{2x+1}\right)+\dfrac{x+2}{x+1}$

$$= \left[\dfrac{(2x-1)(2x+1)-(x+1)(x-1)}{(x-1)(2x+1)}\right]+\dfrac{x+2}{x+1}$$

$$= \dfrac{(4x^2-1)-(x^2-1)}{(x-1)(2x+1)}+\dfrac{x+2}{x+1} = \dfrac{3x^2}{(x-1)(2x+1)}+\dfrac{x+2}{x+1}$$

$$= \dfrac{3x^2(x+1)+(x+2)(x-1)(2x+1)}{(x^2-1)(2x+1)} = \dfrac{5x^3+6x^2-3x-2}{2x^3+x^2-2x-1}$$

Exercise 3.11

1. Simplify the following as a quotient of two polynomials in the simplest form.

(i) $\dfrac{x^3}{x-2}+\dfrac{8}{2-x}$ (ii) $\dfrac{x+2}{x^2+3x+2}+\dfrac{x-3}{x^2-2x-3}$

(iii) $\dfrac{x^2-x-6}{x^2-9}+\dfrac{x^2+2x-24}{x^2-x-12}$ (iv) $\dfrac{x-2}{x^2-7x+10}+\dfrac{x+3}{x^2-2x-15}$

(v) $\frac{2x^2-5x+3}{x^2-3x+2}-\frac{2x^2-7x-4}{2x^2-3x-2}$ (vi) $\frac{x^2-4}{x^2+6x+8}-\frac{x^2-11x+30}{x^2-x-20}$

(vii) $\left[\frac{2x+5}{x+1}+\frac{x^2+1}{x^2-1}\right]-\left(\frac{3x-2}{x-1}\right)$ (viii) $\frac{1}{x^2+3x+2}+\frac{1}{x^2+5x+6}-\frac{2}{x^2+4x+3}$.

2. Which rational expression should be added to $\frac{x^3-1}{x^2+2}$ to get $\frac{3x^3+2x^2+4}{x^2+2}$?

3. Which rational expression should be subtracted from $\frac{4x^3-7x^2+5}{2x-1}$ to get $2x^2-5x+1$?

4. If $P=\frac{x}{x+y}$, $Q=\frac{y}{x+y}$, then find $\frac{1}{P-Q}-\frac{2Q}{P^2-Q^2}$.

3.7 Square root

Let $a\in\mathbb{R}$ be a non negative real number. A square root of a, is a real number b such that $b^2=a$. The positive square root of a is denoted by $\sqrt[2]{a}$ or $\sqrt{a}$. Even though both $(-3)^2=9$ and $(+3)^2=9$ are true, the radical sign $\sqrt{\ }$ is used to indicate the positive square root of the number under it. Hence $\sqrt{9}=3$. Similarly, we have $\sqrt{121}=11$, $\sqrt{10000}=100$.

In the same way, the square root of any expression or a polynomial is an expression whose square is equal to the given expression. In the case of polynomials, we take

$$\sqrt{(p(x))^2}=|p(x)|,\text{ where } |p(x)|=\begin{cases}p(x) & \text{if } p(x)\geq 0\\ -p(x) & \text{if } p(x)<0\end{cases}.$$ For example,

$$\sqrt{(x-a)^2}=|(x-a)| \text{ and } \sqrt{(a-b)^2}=|(a-b)|.$$

In general, the following two methods are very familiar to find the square root of a given polynomial (i) factorization method (ii) division method.

In this section, let us learn the factorization method through some examples for both the expressions and polynomials when they are factorable.

3.7.1 Square root by factorization method

Example 3.31

Find the square root of

(i) $121(x-a)^4(x-b)^6(x-c)^{12}$ (ii) $\frac{81x^4y^6z^8}{64w^{12}s^{14}}$ (iii) $(2x+3y)^2-24xy$

Solution

(i) $\sqrt{121(x-a)^4(x-b)^6(x-c)^{12}}=11|(x-a)^2(x-b)^3(x-c)^6|$

(ii) $\sqrt{\frac{81x^4y^6z^8}{64w^{12}s^{14}}}=\frac{9}{8}\left|\frac{x^2y^3z^4}{w^6s^7}\right|$

(iii) $\sqrt{(2x+3y)^2-24xy}=\sqrt{4x^2+12xy+9y^2-24xy}=\sqrt{(2x-3y)^2}$
$=|(2x-3y)|$

Example 3.32

Find the square root of (i) $4x^2 + 20xy + 25y^2$ (ii) $x^6 + \frac{1}{x^6} - 2$
(iii) $(6x^2 - x - 2)(3x^2 - 5x + 2)(2x^2 - x - 1)$

Solution

(i) $\sqrt{4x^2 + 20xy + 25y^2} = \sqrt{(2x+5y)^2} = |(2x+5y)|$

(ii) $\sqrt{x^6 + \frac{1}{x^6} - 2} = \sqrt{\left(x^3 - \frac{1}{x^3}\right)^2} = \left|\left(x^3 - \frac{1}{x^3}\right)\right|$

(iii) First, let us factorize the polynomials

$6x^2 - x - 2 = (2x+1)(3x-2)$; $3x^2 - 5x + 2 = (3x-2)(x-1)$ and

$2x^2 - x - 1 = (x-1)(2x+1)$

Now, $\sqrt{(6x^2 - x - 2)(3x^2 - 5x + 2)(2x^2 - x - 1)}$

$= \sqrt{(2x+1)(3x-2)\times(3x-2)(x-1)\times(x-1)(2x+1)}$

$= \sqrt{(2x+1)^2(3x-2)^2(x-1)^2} = |(2x+1)(3x-2)(x-1)|$

Exercise 3.12

1. Find the square root of the following

(i) $196a^6b^8c^{10}$ (ii) $289(a-b)^4(b-c)^6$ (iii) $(x+11)^2 - 44x$

(iv) $(x-y)^2 + 4xy$ (v) $121x^8y^6 \div 81x^4y^8$ (vi) $\dfrac{64(a+b)^4(x-y)^8(b-c)^6}{25(x+y)^4(a-b)^6(b+c)^{10}}$

2. Find the square root of the following:

(i) $16x^2 - 24x + 9$

(ii) $(x^2 - 25)(x^2 + 8x + 15)(x^2 - 2x - 15)$

(iii) $4x^2 + 9y^2 + 25z^2 - 12xy + 30yz - 20zx$

(iv) $x^4 + \frac{1}{x^4} + 2$

(v) $(6x^2 + 5x - 6)(6x^2 - x - 2)(4x^2 + 8x + 3)$

(vi) $(2x^2 - 5x + 2)(3x^2 - 5x - 2)(6x^2 - x - 1)$

3.7.2 Finding the square root of a polynomial by division method

In this method, we find the square root of a polynomial which cannot easily be reduced into product of factors. Also division method is a convenient one when the polynomials are of higher degrees.

One can find the square root of a polynomial the same way of finding the square root of a positive integer. Let us explain this method with the following examples.

To find (i) $\sqrt{66564}$

$$
\begin{array}{r|l}
 & \;\;2\;\;\;5\;\;\;8 \\
\hline
2 & 6\;\;65\;\;64 \\
 & 4 \\
\hline
45 & \;\;2\;\;65 \\
 & \;\;2\;\;25 \\
\hline
508 & \;\;\;\;\;40\;\;64 \\
 & \;\;\;\;\;40\;\;64 \\
\hline
 & \;\;\;\;\;\;\;\;0
\end{array}
$$

(ii) $\sqrt{9x^4 + 12x^3 + 10x^2 + 4x + 1}$

Let $p(x) = 9x^4 + 12x^3 + 10x^2 + 4x + 1$

$$
\begin{array}{r|l}
 & 3x^2 \;+\; 2x \;+\; 1 \\
\hline
3x^2 & 9x^4 + 12x^3 + 10x^2 + 4x + 1 \\
 & 9x^4 \\
\hline
6x^2 + 2x & \qquad 12x^3 + 10x^2 \\
 & \qquad 12x^3 + 4x^2 \\
\hline
6x^2 + 4x + 1 & \qquad\qquad 6x^2 + 4x + 1 \\
 & \qquad\qquad 6x^2 + 4x + 1 \\
\hline
 & \qquad\qquad\quad 0
\end{array}
$$

Therefore, $\sqrt{66564} = 258$ and $\sqrt{9x^4 + 12x^3 + 10x^2 + 4x + 1} = |3x^2 + 2x + 1|$

Remarks

(i) While writing the polynomial in **ascending or descending** powers of x, insert zeros for missing terms.

(ii) The above method can be compared with the following procedure.

$$\sqrt{9x^4 + 12x^3 + 10x^2 + 4x + 1} = \sqrt{(a+b+c)^2}$$

Therefore, it is a matter of finding the suitable a, b and c.

Now, $\quad (a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$

$$= a^2 + b^2 + 2ab + 2ac + 2bc + c^2$$

$$= a^2 + (2a+b)b + (2a+2b+c)c$$

$$= (3x^2)^2 + (6x^2 + 2x)(2x) + (6x^2 + 4x + 1)(1)$$

Thus, $\sqrt{9x^4 + 12x^3 + 10x^2 + 4x + 1} = |3x^2 + 2x + 1|$, where $a = 3x^2$, $b = 2x$ and $c = 1$

Aliter : To find the square root, first write $9x^4 + 12x^3 + 10x^2 + 4x + 1$

$$= (mx^2 + nx + l)^2 = m^2x^4 + 2mnx^3 + (n^2 + 2lm)x^2 + 2nlx + l^2$$

Compare the coefficients and then find the suitable constants m, n, l.

(iii) It is also quite interesting to note the following :

$$25x^4 - 30x^3 + 29x^2 - 12x + 4 = 25x^4 - 30x^3 + 9x^2 + 20x^2 - 12x + 4$$

$$= (5x^2)^2 + [10x^2 + (-3x)](-3x) + (10x^2 - 6x + 2)2$$

$$= (5x^2)^2 + [2(5x^2) + (-3x)](-3x) + [2(5x^2) + 2(-3x) + 2]2$$

$$= a^2 + [2a + (-b)](-b) + [2a + 2(-b) + c]c$$

$$= a^2 + (-b)^2 + c^2 + 2a(-b) + 2(-b)c + 2ac$$

$$= (a - b + c)^2, \quad \text{where } a = 5x^2,\ b = 3x,\ c = 2$$

$\therefore \quad \sqrt{25x^4 - 30x^3 + 29x^2 - 12x + 4} = |5x^2 - 3x + 2|$.

Example 3.33

Find the square root of $x^4 - 10x^3 + 37x^2 - 60x + 36$.

Solution Given polynomial is already in descending powers of x.

$$
\begin{array}{r|r}
 & x^2 - 5x + 6 \\
\hline
x^2 & x^4 - 10x^3 + 37x^2 - 60x + 36 \\
 & x^4 \\
\hline
2x^2 - 5x & -10x^3 + 37x^2 \\
 & -10x^3 + 25x^2 \\
\hline
2x^2 - 10x + 6 & 12x^2 - 60x + 36 \\
 & 12x^2 - 60x + 36 \\
\hline
 & 0
\end{array}
$$

Thus, $\sqrt{x^4 - 10x^3 + 37x^2 - 60x + 36} = |(x^2 - 5x + 6)|$

Example 3.34

Find the square root of $x^4 - 6x^3 + 19x^2 - 30x + 25$

Solution Let us write the polynomial in ascending powers of x and find the square root.

$$
\begin{array}{r|r}
 & 5 - 3x + x^2 \\
\hline
5 & 25 - 30x + 19x^2 - 6x^3 + x^4 \\
 & 25 \\
\hline
10-3x & -30x + 19x^2 \\
 & -30x + 9x^2 \\
\hline
10 - 6x + x^2 & 10x^2 - 6x^3 + x^4 \\
 & 10x^2 - 6x^3 + x^4 \\
\hline
 & 0
\end{array}
$$

Hence, the square root of the given polynomial is $|x^2 - 3x + 5|$

Example 3.35

If $m - nx + 28x^2 + 12x^3 + 9x^4$ is a perfect square, then find the values of m and n.

Solution Arrange the polynomial in descending power of x.

$$9x^4 + 12x^3 + 28x^2 - nx + m.$$

Now,

$$
\begin{array}{r|l}
 & 3x^2 + 2x + 4 \\
\hline
3x^2 & 9x^4 + 12x^3 + 28x^2 - nx + m \\
 & 9x^4 \\
\hline
6x^2 + 2x & \quad 12x^3 + 28x^2 \\
 & \quad 12x^3 + 4x^2 \\
\hline
6x^2 + 4x + 4 & \qquad 24x^2 - nx + m \\
 & \qquad 24x^2 + 16x + 16 \\
\hline
 & \qquad\qquad 0
\end{array}
$$

Since the given polynomial is a perfect square, we must have $n = -16$ and $m = 16$.

Exercise 3.13

1. Find the square root of the following polynomials by division method.
   (i) $x^4 - 4x^3 + 10x^2 - 12x + 9$ (ii) $4x^4 + 8x^3 + 8x^2 + 4x + 1$
   (iii) $9x^4 - 6x^3 + 7x^2 - 2x + 1$ (iv) $4 + 25x^2 - 12x - 24x^3 + 16x^4$
2. Find the values of a and b if the following polynomials are perfect squares.
   (i) $4x^4 - 12x^3 + 37x^2 + ax + b$ (ii) $x^4 - 4x^3 + 10x^2 - ax + b$
   (iii) $ax^4 + bx^3 + 109x^2 - 60x + 36$ (iv) $ax^4 - bx^3 + 40x^2 + 24x + 36$

3.8 Quadratic equations

Greek mathematician Euclid developed a geometrical approach for finding out lengths which in our present day terminology, are solutions of quadratic equations. Solving quadratic equations in general form is often credited to ancient Indian Mathematicians. In fact, Brahma Gupta (A.D 598 - 665) gave an explicit formula to solve a quadratic equation of the form $ax^2 + bx = c$. Later *Sridhar Acharya* (1025 A.D) derived a formula, now known as the quadratic formula, (as quoted by Bhaskara II) for solving a quadratic equation by the method of completing the square.

In this section, we will learn solving quadratic equations, by various methods. We shall also see some applications of quadratic equations.

Definition

A quadratic equation in the variable x is an equation of the form $ax^2 + bx + c = 0$, where a, b, c are real numbers and $a \neq 0$.

In fact, any equation of the form $p(x) = 0$, where $p(x)$ is a polynomial of degree 2, is a quadratic equation, whose standard form is $ax^2 + bx + c = 0,\ a \neq 0$.

For example, $2x^2 - 3x + 4 = 0,\ 1 - x + x^2 = 0$ are some quadratic equations.

3.8.1 Solution of a quadratic equation by factorization method

Factorization method can be used when the quadratic equation can be factorized into linear factors. Given a product, if any factor is zero, then the whole product is zero. Conversely, if a product is equal to zero, then some factor of that product must be zero, and any factor which contains an unknown may be equal to zero. Thus, in solving a quadratic equation, we find the values of x which make each of the factors zero. That is, we may equate each factor to zero and solve for the unknown.

Example 3.36

Solve $6x^2 - 5x - 25 = 0$

Solution Given $6x^2 - 5x - 25 = 0$.

First, let us find α and β such that $\alpha + \beta = -5$ and $\alpha\beta = 6 \times(-25) = -150$, where -5 is the coefficient of x. Thus, we get $\alpha = -15$ and $\beta = 10$.

Next, $6x^2 - 5x - 25 = 6x^2 - 15x + 10x - 25 = 3x(2x - 5) + 5(2x - 5)$

$$= (2x - 5)(3x + 5).$$

Therefore, the solution set is obtained from $2x - 5 = 0$ and $3x + 5 = 0$

Thus, $x = \frac{5}{2}$, $x = -\frac{5}{3}$.

Hence, solution set is $\left\{-\frac{5}{3}, \frac{5}{2}\right\}$.

Example 3.37

Solve $\frac{6}{7x - 21} - \frac{1}{x^2 - 6x + 9} + \frac{1}{x^2 - 9} = 0$

Solution Given equation appears to be a non-quadratic equation. But when we simplify the equation, it will reduce to a quadratic equation.

Now, $\frac{6}{7(x-3)} - \frac{1}{(x-3)^2} + \frac{1}{(x+3)(x-3)} = 0$

$$\Longrightarrow \frac{6(x^2 - 9) - 7(x + 3) + 7(x - 3)}{7(x - 3)^2(x + 3)} = 0$$

$$\Longrightarrow 6x^2 - 54 - 42 = 0 \Longrightarrow x^2 - 16 = 0$$

The equation $x^2 = 16$ is quadratic and hence we have two values $x = 4$ and $x = -4$.

$\therefore$ Solution set is $\{-4, 4\}$

Example 3.38

Solve $\sqrt{24 - 10x} = 3 - 4x$, $3 - 4x > 0$

Solution Given $\sqrt{24 - 10x} = 3 - 4x$

Squaring on both sides, we get, $24 - 10x = (3 - 4x)^2$

$$\Longrightarrow 16x^2 - 14x - 15 = 0 \qquad \Longrightarrow 16x^2 - 24x + 10x - 15 = 0$$

$$\Longrightarrow \quad (8x+5)(2x-3)=0 \text{ which gives } x = \frac{3}{2} \text{ or } -\frac{5}{8}$$

When $x = \frac{3}{2}$, $3-4x = 3-4\left(\frac{3}{2}\right) < 0$ and hence, $x = \frac{3}{2}$ is not a solution of the equation.

When $x = -\frac{5}{8}$, $3-4x > 0$ and hence, the solution set is $\left\{-\frac{5}{8}\right\}$.

Remarks

To solve radical equation like the above, we rely on the **squaring property** :

$a = b \Longrightarrow a^2 = b^2$. Unfortunately, this squaring property does not guarantee that all solutions of the new equation are solutions of the original equation. For example, on squaring the equation $x = 5$ we get $x^2 = 25$, which in turn gives $x = 5$ and $x = -5$. But $x = -5$ is not a solution of the original equation. Such a solution is called an **extraneous** solution.

Thus, the above example shows that when squaring on both sides of a radical equation, the solution of the final equation must be checked to determine whether they are solutions of the original equation or not. This is necessary because no solution of the original equation will be lost by squaring but certain values may be introduced which are roots of the new equation but not of the original equation.

Exercise 3.14

Solve the following quadratic equations by factorization method.

(i) $(2x+3)^2 - 81 = 0$ (ii) $3x^2 - 5x - 12 = 0$ (iii) $\sqrt{5}x^2 + 2x - 3\sqrt{5} = 0$

(iv) $3(x^2 - 6) = x(x+7) - 3$ (v) $3x - \frac{8}{x} = 2$ (vi) $x + \frac{1}{x} = \frac{26}{5}$

(vii) $\frac{x}{x+1} + \frac{x+1}{x} = \frac{34}{15}$ (viii) $a^2b^2x^2 - (a^2+b^2)x + 1 = 0$

(ix) $2(x+1)^2 - 5(x+1) = 12$ (x) $3(x-4)^2 - 5(x-4) = 12$

3.8.2 Solution of a quadratic equation by completing square

From $\left(x + \frac{b}{2}\right)^2 = x^2 + bx + \left(\frac{b}{2}\right)^2$, note that the last term $\left(\frac{b}{2}\right)^2$ is the square of half the coefficient of x. Hence, the $x^2 + bx$ lacks only the term $\left(\frac{b}{2}\right)^2$ of being the square of $x + \frac{b}{2}$. Thus, if the square of half the coefficient of x be added to an expression of the form $x^2 + bx$, the result is the square of a binomial. Such an addition is usually known as completing the square. In this section, we shall find the solution of a quadratic equation by the method of completing the square through the following steps.

***Step* 1** If the coefficient of x^2 is 1, go to step 2. If not, divide both sides of the equation by the coefficient of x^2. Get all the terms with variable on one side of equation.

***Step* 2** Find half the coefficient of x and square it. Add this number to both sides of the equation. To solve the equation, use the square root property:

$$x^2 = t \Longrightarrow x = \sqrt{t} \text{ or } x = -\sqrt{t} \text{ where } t \text{ is non-negative.}$$

Example 3.39

Solve the quadratic equation $5x^2 - 6x - 2 = 0$ by completing the square.

Solution Given quadratic equation is $5x^2 - 6x - 2 = 0$

$$\Longrightarrow \quad x^2 - \frac{6}{5}x - \frac{2}{5} = 0 \qquad \text{(Divide on both sides by 5)}$$

$$\Longrightarrow \quad x^2 - 2\left(\frac{3}{5}\right)x = \frac{2}{5} \qquad \left(\frac{3}{5} \text{ is the half of the coefficient of } x\right)$$

$$\Longrightarrow \quad x^2 - 2\left(\frac{3}{5}\right)x + \frac{9}{25} = \frac{9}{25} + \frac{2}{5} \qquad \left(\text{add } \left(\frac{3}{5}\right)^2 = \frac{9}{25} \text{ on both sides}\right)$$

$$\Longrightarrow \quad \left(x - \frac{3}{5}\right)^2 = \frac{19}{25}$$

$$\Longrightarrow \quad x - \frac{3}{5} = \pm\sqrt{\frac{19}{25}} \qquad \text{(take square root on both sides)}$$

Thus, we have $x = \frac{3}{5} \pm \frac{\sqrt{19}}{5} = \frac{3 \pm \sqrt{19}}{5}$.

Hence, the solution set is $\left\{\frac{3+\sqrt{19}}{5}, \frac{3-\sqrt{19}}{5}\right\}$.

Example 3.40

Solve the equation $a^2x^2 - 3abx + 2b^2 = 0$ by completing the square

Solution There is nothing to prove if $a = 0$. For $a \neq 0$, we have

$$a^2x^2 - 3abx + 2b^2 = 0$$

$$\Longrightarrow \quad x^2 - \frac{3b}{a}x + \frac{2b^2}{a^2} = 0 \qquad \Longrightarrow \quad x^2 - 2\left(\frac{3b}{2a}\right)x = \frac{-2b^2}{a^2}$$

$$\Longrightarrow \quad x^2 - 2\left(\frac{3b}{2a}\right)x + \frac{9b^2}{4a^2} = \frac{9b^2}{4a^2} - \frac{2b^2}{a^2}$$

$$\Longrightarrow \quad \left(x - \frac{3b}{2a}\right)^2 = \frac{9b^2 - 8b^2}{4a^2} \qquad \Longrightarrow \quad \left(x - \frac{3b}{2a}\right)^2 = \frac{b^2}{4a^2}$$

$$\Longrightarrow \quad x - \frac{3b}{2a} = \pm\frac{b}{2a} \qquad \Longrightarrow \quad x = \frac{3b \pm b}{2a}$$

Therefore, the solution set is $\left\{\frac{b}{a}, \frac{2b}{a}\right\}$.

3.8.3 Solution of quadratic equation by formula method

In this section, we shall derive the quadratic formula, which is useful for finding the roots of a quadratic equation. Consider a quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$. We rewrite the given equation as

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

$$\Longrightarrow \quad x^2 + 2\left(\frac{b}{2a}\right)x + \frac{c}{a} = 0 \qquad \Longrightarrow \quad x^2 + 2\left(\frac{b}{2a}\right)x = -\frac{c}{a}$$

Adding $\left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}$ both sides we get, $x^2 + 2\left(\frac{b}{2a}\right)x + \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$

That is, $\left(x+\frac{b}{2a}\right)^2 = \frac{b^2-4ac}{4a^2}$

$$\Longrightarrow \quad x+\frac{b}{2a} = \pm\sqrt{\frac{b^2-4ac}{4a^2}} = \pm\frac{\sqrt{b^2-4ac}}{2a}$$

So, we have $x = \dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$ (1)

The solution set is $\left\{\dfrac{-b+\sqrt{b^2-4ac}}{2a}, \dfrac{-b-\sqrt{b^2-4ac}}{2a}\right\}$.

The formula given in equation (1) is known as quadratic formula.

Now, let us solve some quadratic equations using quadratic formula.

Example 3.41

Solve the equation $\frac{1}{x+1}+\frac{2}{x+2}=\frac{4}{x+4}$, where $x+1\neq 0$, $x+2\neq 0$ and $x+4\neq 0$ using quadratic formula.

Solution Note that the given equation is not in the standard form of a quadratic equation.

Consider $\frac{1}{x+1}+\frac{2}{x+2} = \frac{4}{x+4}$

That is, $\frac{1}{x+1} = 2\left[\frac{2}{x+4}-\frac{1}{x+2}\right] = 2\left[\frac{2x+4-x-4}{(x+4)(x+2)}\right]$

$$\frac{1}{x+1} = 2\left[\frac{x}{(x+2)(x+4)}\right]$$

$$x^2+6x+8 = 2x^2+2x$$

Thus, we have $x^2-4x-8 = 0$, which is a quadratic equation.

(The above equation can also be obtained by taking LCM)

Using the quadratic formula we get,

$$x = \frac{4\pm\sqrt{16-4(1)(-8)}}{2(1)} = \frac{4\pm\sqrt{48}}{2}$$

Thus, $x = 2+2\sqrt{3}$ or $2-2\sqrt{3}$

Hence, the solution set is $\{2-2\sqrt{3}, 2+2\sqrt{3}\}$

Exercise 3.15

1 Solve the following quadratic equations by completing the square .

(i) $x^2+6x-7=0$

(ii) $x^2+3x+1=0$

(iii) $2x^2+5x-3=0$

(iv) $4x^2+4bx-(a^2-b^2)=0$

(v) $x^2-(\sqrt{3}+1)x+\sqrt{3}=0$

(vi) $\frac{5x+7}{x-1}=3x+2$

2. Solve the following quadratic equations using quadratic formula.

(i) $x^2 - 7x + 12 = 0$ (ii) $15x^2 - 11x + 2 = 0$

(iii) $x + \frac{1}{x} = 2\frac{1}{2}$ (iv) $3a^2x^2 - abx - 2b^2 = 0$

(v) $a(x^2 + 1) = x(a^2 + 1)$ (vi) $36x^2 - 12ax + (a^2 - b^2) = 0$

(vii) $\frac{x-1}{x+1} + \frac{x-3}{x-4} = \frac{10}{3}$ (viii) $a^2x^2 + (a^2 - b^2)x - b^2 = 0$

3.8.4 Solution of problems involving quadratic equations

In this section, we will solve some simple problems expressed in words and some problems describing day-to-day life situations involving quadratic equation. First we shall form an equation translating the given statement and then solve it. Finally, we choose the solution that is relevant to the given problem.

Example 3.42

The sum of a number and its reciprocal is $5\frac{1}{5}$. Find the number.

Solution Let x denote the required number. Then its reciprocal is $\frac{1}{x}$

By the given condition, $x + \frac{1}{x} = 5\frac{1}{5} \implies \frac{x^2+1}{x} = \frac{26}{5}$

So, $5x^2 - 26x + 5 = 0$

$$\implies 5x^2 - 25x - x + 5 = 0$$

That is, $(5x - 1)(x - 5) = 0 \implies x = 5$ or $\frac{1}{5}$

Thus, the required numbers are $5, \frac{1}{5}$.

Example 3.43

The base of a triangle is 4cm longer than its altitude. If the area of the triangle is 48 sq. cm, then find its base and altitude.

Solution Let the altitude of the triangle be x cm.

By the given condition, the base of the triangle is $(x + 4)$ cm.

Now, the area of the triangle = $\frac{1}{2}(\text{base}) \times \text{height}$

By the given condition $\frac{1}{2}(x + 4)(x) = 48$

$$\implies x^2 + 4x - 96 = 0 \implies (x + 12)(x - 8) = 0$$

$$\implies x = -12 \text{ or } 8$$

But $x = -12$ is not possible (since the length should be positive)

Therefore, $x = 8$ and hence, $x + 4 = 12$.

Thus, the altitude of the triangle is 8cm and the base of the triangle is 12cm.

Example 3.44

A car left 30 minutes later than the scheduled time. In order to reach its destination 150 km away in time, it has to increase its speed by 25 km/hr from its usual speed. Find its usual speed.

Solution Let the usual speed of the car be x km/hr.

Thus, the increased speed of the car is $(x+25)$ km/hr

Total distance $=150$ km; Time taken $=\dfrac{\text{Distance}}{\text{Speed}}$.

Let T_1 and T_2 be the time taken in hours by the car to cover the given distance in scheduled time and decreased time (as the speed is increased) respectively.

By the given information $T_1 - T_2 = \frac{1}{2}$ hr (30 minutes $= \frac{1}{2}$ hr)

$$\implies \frac{150}{x} - \frac{150}{x+25} = \frac{1}{2} \implies 150\left[\frac{x+25-x}{x(x+25)}\right] = \frac{1}{2}$$

$$\implies x^2 + 25x - 7500 = 0 \implies (x+100)(x-75) = 0$$

Thus, $x = 75$ or -100, but $x = -100$ is not an admissible value.

Therefore, the usual speed of the car is 75 km/hr.

Exercise 3.16

1. The sum of a number and its reciprocal is $\frac{65}{8}$. Find the number.
2. The difference of the squares of two positive numbers is 45. The square of the smaller number is four times the larger number. Find the numbers.
3. A farmer wishes to start a 100 sq.m rectangular vegetable garden. Since he has only 30 m barbed wire, he fences the sides of the rectangular garden letting his house compound wall act as the fourth side fence. Find the dimension of the garden.
4. A rectangular field is 20 m long and 14 m wide. There is a path of equal width all around it having an area of 111 sq. metres. Find the width of the path on the outside.
5. A train covers a distance of 90 km at a uniform speed. Had the speed been 15 km/hr more, it would have taken 30 minutes less for the journey. Find the original speed of the train.
6. The speed of a boat in still water is 15 km/hr. It goes 30 km upstream and return downstream to the original point in 4 hrs 30 minutes. Find the speed of the stream.
7. One year ago, a man was 8 times as old as his son. Now his age is equal to the square of his son's age. Find their present ages.
8. A chess board contains 64 equal squares and the area of each square is 6.25 cm^2. A border around the board is 2 cm wide. Find the length of the side of the chess board.

9. A takes 6 days less than the time taken by B to finish a piece of work. If both A and B together can finish it in 4 days, find the time that B would take to finish this work by himself.

10. Two trains leave a railway station at the same time. The first train travels due west and the second train due north. The first train travels 5 km/hr faster than the second train. If after two hours, they are 50 km apart, find the average speed of each train.

3.8.5 Nature of roots of a quadratic equation

The roots of the equation $ax^2 + bx + c = 0$ are given by $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

If $b^2 - 4ac > 0$, we get two distinct real roots

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

If $b^2 - 4ac = 0$, then the equation has two equal roots $x = \dfrac{-b}{2a}$.

If $b^2 - 4ac < 0$, then $\sqrt{b^2 - 4ac}$ is not a real number. Therefore there is no real root for the given quadratic equation.

So, evidently the nature of roots depends on the values of $b^2 - 4ac$. The value of the expression $b^2 - 4ac$ discriminates the nature of the roots of $ax^2 + bx + c = 0$ and so it is called the discriminant of the quadratic equation and denoted by the symbol $\triangle$.

Discriminant $\triangle = b^2 - 4ac$	Nature of roots
$\triangle > 0$	Real and unequal
$\triangle = 0$	Real and equal.
$\triangle < 0$	No real roots. (It has imaginary roots)

Example 3.45

Determine the nature of roots of the following quadratic equations

(i) $x^2 - 11x - 10 = 0$ (ii) $4x^2 - 28x + 49 = 0$ (iii) $2x^2 + 5x + 5 = 0$

Solution For $ax^2 + bx + c = 0$, the discriminant, $\triangle = b^2 - 4ac$.

(i) Here, $a = 1$; $b = -11$ and $c = -10$.

Now, the discriminant is $\triangle = b^2 - 4ac$

$$= (-11)^2 - 4(1)(-10) = 121 + 40 = 161$$

Thus, $\triangle > 0$. Therefore, the roots are real and unequal.

(ii) Here, $a = 4$, $b = -28$ and $c = 49$.

Now, the discriminant is $\triangle = b^2 - 4ac$

$$= (-28)^2 - 4(4)(49) = 0$$

Since $\triangle = 0$, the roots of the given equation are real and equal.

(iii) Here, $a = 2$, $b = 5$ and $c = 5$.

Now, the discriminant $\triangle = b^2 - 4ac$

$= (5)^2 - 4(2)(5)$

$= 25 - 40 = -15$

Since $\triangle < 0$, the equation has no real roots.

Example 3.46

Prove that the roots of the equation $(a-b+c)x^2 + 2(a-b)x + (a-b-c) = 0$ are rational numbers for all real numbers a and b and for all rational c.

Solution Let the given equation be of the form $Ax^2 + Bx + C = 0$. Then,

$A = a-b+c$, $B = 2(a-b)$ and $C = a-b-c$.

Now, the discriminant of $Ax^2 + Bx + C = 0$ is

$$
\begin{aligned}
B^2 - 4AC &= [2(a-b)]^2 - 4(a-b+c)(a-b-c) \\
&= 4(a-b)^2 - 4[(a-b)+c][(a-b)-c] \\
&= 4(a-b)^2 - 4[(a-b)^2 - c^2] \\
\triangle &= 4(a-b)^2 - 4(a-b)^2 + 4c^2 = 4c^2, \text{ a perfect square.}
\end{aligned}
$$

Therefore, $\triangle > 0$ and it is a perfect square.

Hence, the roots of the given equation are rational numbers.

Example 3.47

Find the values of k so that the equation $x^2 - 2x(1+3k) + 7(3+2k) = 0$ has real and equal roots.

Solution The given equation is $x^2 - 2x(1+3k) + 7(3+2k) = 0$. (1)

Let the equation (1) be in the form $ax^2 + bx + c = 0$

Here, $a = 1$, $b = -2(3k+1)$, $c = 7(3+2k)$.

Now, the discriminant is $\triangle = b^2 - 4ac$

$= (-2(3k+1))^2 - 4(1)(7)(3+2k)$

$= 4(9k^2 + 6k + 1) - 28(3+2k) = 4(9k^2 - 8k - 20)$

Given that the equation has equal roots. Thus, $\triangle = 0$

$\Longrightarrow 9k^2 - 8k - 20 = 0$

$\Longrightarrow (k-2)(9k+10) = 0$

Thus, $k = 2, -\frac{10}{9}$.

Exercise 3.17

1. Determine the nature of the roots of the equation.

(i) $x^2 - 8x + 12 = 0$ (ii) $2x^2 - 3x + 4 = 0$

(iii) $9x^2 + 12x + 4 = 0$ (iv) $3x^2 - 2\sqrt{6}x + 2 = 0$

(v) $\frac{3}{5}x^2 - \frac{2}{3}x + 1 = 0$ (vi) $(x - 2a)(x - 2b) = 4ab$

2. Find the values of k for which the roots are real and equal in each of the following equations.

(i) $2x^2 - 10x + k = 0$ (ii) $12x^2 + 4kx + 3 = 0$

(iii) $x^2 + 2k(x - 2) + 5 = 0$ (iv) $(k + 1)x^2 - 2(k - 1)x + 1 = 0$

3. Show that the roots of the equation $x^2 + 2(a + b)x + 2(a^2 + b^2) = 0$ are unreal.

4. Show that the roots of the equation $3p^2x^2 - 2pqx + q^2 = 0$ are not real.

5. If the roots of the equation $(a^2 + b^2)x^2 - 2(ac + bd)x + c^2 + d^2 = 0$, where a, b, c and $d \neq 0$, are equal, prove that $\frac{a}{b} = \frac{c}{d}$.

6. Show that the roots of the equation $(x - a)(x - b) + (x - b)(x - c) + (x - c)(x - a) = 0$ are always real and they cannot be equal unless $a = b = c$.

7. If the equation $(1 + m^2)x^2 + 2mcx + c^2 - a^2 = 0$ has equal roots, then prove that $c^2 = a^2(1 + m^2)$.

3.8.6 Relations between roots and coefficients of a quadratic equation

Consider a quadratic equation $ax^2 + bx + c = 0$, where a , b , c are real numbers and $a \neq 0$. The roots of the given equation are α and β, where

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Then, the sum of the roots,

$$\alpha + \beta = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-b}{a} = \frac{-\text{coefficient of } x}{\text{coefficient of } x^2}$$

and the product of roots,

$$\alpha\beta = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \times \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{b^2 - (b^2 - 4ac)}{4a^2} = \frac{4ac}{4a^2}$$

$$= \frac{c}{a} = \frac{\text{constant term}}{\text{coefficient of } x^2}$$

Therefore, if α, β are the roots of $ax^2 + bx + c = 0$, then

(i) the sum of the roots, $\alpha + \beta = -\frac{b}{a}$

(ii) the product of roots, $\alpha\beta = \frac{c}{a}$

Formation of quadratic equation when roots are given

Let α and β be the roots of a quadratic equation.

Then $(x - \alpha)$ and $(x - \beta)$ are factors.

$\therefore \quad (x - \alpha)(x - \beta) = 0$

$\Longrightarrow \quad x^2 - (\alpha + \beta)x + \alpha\beta = 0$

That is, $x^2 - (\text{sum of roots})x + \text{product of roots} = 0$

> *Note*
>
> There are infinitely many quadratic equations with the same roots.

Example 3.48

If one of the roots of the equation $3x^2 - 10x + k = 0$ is $\frac{1}{3}$, then find the other root and also the value of k.

Solution The given equation is $3x^2 - 10x + k = 0$.

Let the two roots be α and β.

$$\therefore \quad \alpha + \beta = \frac{-(-10)}{3} = \frac{10}{3} \qquad (1)$$

Substituting $\alpha = \frac{1}{3}$ in (1) we get $\beta = 3$

Also, $\alpha\beta = \frac{k}{3}, \quad \Longrightarrow \quad k = 3$

Thus, the other root $\beta = 3$ and the value of $k = 3$.

Example 3.49

If the sum and product of the roots of the quadratic equation $ax^2 - 5x + c = 0$ are both equal to 10, then find the values of a and c.

Solution The given equation is $ax^2 - 5x + c = 0$.

Sum of the roots, $\frac{5}{a} = 10, \quad \Longrightarrow \quad a = \frac{1}{2}$

Product of the roots, $\frac{c}{a} = 10$

$\Longrightarrow \quad c = 10a = 10 \times \frac{1}{2} = 5$

Hence, $a = \frac{1}{2}$ and $c = 5$

Note

If α and β are the roots of $ax^2+bx+c=0$, then many expressions in α and β like $\alpha^2+\beta^2$, $\alpha^2\beta^2$, $\alpha^2-\beta^2$ etc., can be evaluated using the values of $\alpha+\beta$ and $\alpha\beta$.

Let us write some results involving α and β.

(i) $|\alpha-\beta| = \sqrt{(\alpha+\beta)^2-4\alpha\beta}$

(ii) $\alpha^2+\beta^2 = [(\alpha+\beta)^2-2\alpha\beta]$

(iii) $\alpha^2-\beta^2 = (\alpha+\beta)(\alpha-\beta) = (\alpha+\beta)[\sqrt{(\alpha+\beta)^2-4\alpha\beta}]$ only if $\alpha\geq\beta$

(iv) $\alpha^3+\beta^3 = (\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)$

(v) $\alpha^3-\beta^3 = (\alpha-\beta)^3+3\alpha\beta(\alpha-\beta)$

(vi) $\alpha^4+\beta^4 = (\alpha^2+\beta^2)^2-2\alpha^2\beta^2 = [(\alpha+\beta)^2-2\alpha\beta]^2-2(\alpha\beta)^2$

(vii) $\alpha^4-\beta^4 = (\alpha+\beta)(\alpha-\beta)(\alpha^2+\beta^2)$

Example 3.50

If α and β are the roots of the equation $2x^2-3x-1=0$, find the values of

(i) $\alpha^2+\beta^2$ (ii) $\dfrac{\alpha}{\beta}+\dfrac{\beta}{\alpha}$

(iii) $\alpha-\beta$ if $\alpha>\beta$ (iv) $\left(\dfrac{\alpha^2}{\beta}+\dfrac{\beta^2}{\alpha}\right)$

(v) $\left(\alpha+\dfrac{1}{\beta}\right)\left(\dfrac{1}{\alpha}+\beta\right)$ (vi) $\alpha^4+\beta^4$ (vii) $\dfrac{\alpha^3}{\beta}+\dfrac{\beta^3}{\alpha}$

Solution Given equation is $2x^2-3x-1=0$

Let the given equation be written as $ax^2+bx+c=0$

Then, $a=2$, $b=-3$, $c=-1$. Given α and β are the roots of the equation.

$\therefore\ \alpha+\beta = \dfrac{-b}{a} = \dfrac{-(-3)}{2} = \dfrac{3}{2}$ and $\alpha\beta = -\dfrac{1}{2}$

(i) $\alpha^2+\beta^2 = (\alpha+\beta)^2-2\alpha\beta = \left(\dfrac{3}{2}\right)^2-2\left(-\dfrac{1}{2}\right) = \dfrac{9}{4}+1 = \dfrac{13}{4}$

(ii) $\dfrac{\alpha}{\beta}+\dfrac{\beta}{\alpha} = \dfrac{\alpha^2+\beta^2}{\alpha\beta} = \dfrac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta} = \dfrac{\left(\frac{3}{2}\right)^2-2\left(-\frac{1}{2}\right)}{-\frac{1}{2}} = \dfrac{13}{4}\times(-2) = -\dfrac{13}{2}$

(iii) $\alpha-\beta = \sqrt{(\alpha+\beta)^2-4\alpha\beta}$

$= \left[\left(\dfrac{3}{2}\right)^2-4\times\left(-\dfrac{1}{2}\right)\right]^{\frac{1}{2}} = \left(\dfrac{9}{4}+2\right)^{\frac{1}{2}} = \dfrac{\sqrt{17}}{2}$

(iv) $\dfrac{\alpha^2}{\beta}+\dfrac{\beta^2}{\alpha}=\dfrac{\alpha^3+\beta^3}{\alpha\beta}=\dfrac{(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)}{\alpha\beta}=\dfrac{\frac{27}{8}+\frac{9}{4}}{\frac{-1}{2}}=-\dfrac{45}{4}$

(v) $\left(\alpha+\dfrac{1}{\beta}\right)\left(\dfrac{1}{\alpha}+\beta\right)=\dfrac{(\alpha\beta+1)(1+\alpha\beta)}{\alpha\beta}$

$$=\frac{(1+\alpha\beta)^2}{\alpha\beta}=\frac{\left(1-\frac{1}{2}\right)^2}{-\frac{1}{2}}=-\frac{1}{2}$$

(vi) $\alpha^4+\beta^4=(\alpha^2+\beta^2)^2-2\alpha^2\beta^2$

$$=\left(\frac{13}{4}\right)^2-2\left(-\frac{1}{2}\right)^2=\left(\frac{169}{16}-\frac{1}{2}\right)=\frac{161}{16}.$$

(vii) $\dfrac{\alpha^3}{\beta}+\dfrac{\beta^3}{\alpha}=\dfrac{\alpha^4+\beta^4}{\alpha\beta}=\left(\dfrac{161}{16}\right)\left(-\dfrac{2}{1}\right)=-\dfrac{161}{8}.$

Example 3.51

Form the quadratic equation whose roots are $7+\sqrt{3}$ and $7-\sqrt{3}$.

Solution Given roots are $7+\sqrt{3}$ and $7-\sqrt{3}$.

$\therefore$ Sum of the roots $=7+\sqrt{3}+7-\sqrt{3}=14$.

Product of roots $=(7+\sqrt{3})(7-\sqrt{3})=(7)^2-(\sqrt{3})^2=49-3=46$.

The required equation is $x^2-(\text{sum of the roots})x+(\text{product of the roots})=0$

Thus, the required equation is $x^2-14x+46=0$

Example 3.52

If α and β are the roots of the equation

$3x^2-4x+1=0$, form a quadratic equation whose roots are $\dfrac{\alpha^2}{\beta}$ and $\dfrac{\beta^2}{\alpha}$.

Solution Since α,β are the roots of the equation $3x^2-4x+1=0$,

we have $\alpha+\beta=\dfrac{4}{3}$, $\quad \alpha\beta=\dfrac{1}{3}$

Now, for the required equation, the sum of the roots $=\left(\dfrac{\alpha^2}{\beta}+\dfrac{\beta^2}{\alpha}\right)=\dfrac{\alpha^3+\beta^3}{\alpha\beta}$

$$=\frac{(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)}{\alpha\beta}=\frac{\left(\frac{4}{3}\right)^3-3\times\frac{1}{3}\times\frac{4}{3}}{\frac{1}{3}}=\frac{28}{9}$$

Also, product of the roots $=\left(\dfrac{\alpha^2}{\beta}\right)\left(\dfrac{\beta^2}{\alpha}\right)=\alpha\beta=\dfrac{1}{3}$

$\therefore$ The required equation is $x^2-\dfrac{28}{9}x+\dfrac{1}{3}=0$ or $9x^2-28x+3=0$

Exercise 3.18

1. Find the sum and the product of the roots of the following equations.

   (i) $x^2 - 6x + 5 = 0$ (ii) $kx^2 + rx + pk = 0$

   (iii) $3x^2 - 5x = 0$ (iv) $8x^2 - 25 = 0$

2. Form a quadratic equation whose roots are

   (i) $3, 4$ (ii) $3+\sqrt{7}, 3-\sqrt{7}$ (iii) $\frac{4+\sqrt{7}}{2}, \frac{4-\sqrt{7}}{2}$

3. If α and β are the roots of the equation $3x^2 - 5x + 2 = 0$, then find the values of

   (i) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$ (ii) $\alpha - \beta$ (iii) $\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha}$

4. If α and β are the roots of the equation $3x^2 - 6x + 4 = 0$, find the value of $\alpha^2 + \beta^2$.

5. If α, β are the roots of $2x^2 - 3x - 5 = 0$, form a equation whose roots are α^2 and β^2.

6. If α, β are the roots of $x^2 - 3x + 2 = 0$, form a quadratic equation whose roots are $-\alpha$ and $-\beta$.

7. If α and β are the roots of $x^2 - 3x - 1 = 0$, then form a quadratic equation whose roots are $\frac{1}{\alpha^2}$ and $\frac{1}{\beta^2}$.

8. If α and β are the roots of the equation $3x^2 - 6x + 1 = 0$, form an equation whose roots are (i) $\frac{1}{\alpha}, \frac{1}{\beta}$ (ii) $\alpha^2\beta, \beta^2\alpha$ (iii) $2\alpha+\beta, 2\beta+\alpha$

9. Find a quadratic equation whose roots are the reciprocal of the roots of the equation $4x^2 - 3x - 1 = 0$.

10. If one root of the equation $3x^2 + kx - 81 = 0$ is the square of the other, find k.

11. If one root of the equation $2x^2 - ax + 64 = 0$ is twice the other, then find the value of a

12. If α and β are the roots of $5x^2 - px + 1 = 0$ and $\alpha - \beta = 1$, then find p.

Exercise 3.19

Choose the correct answer.

1. If the system $6x - 2y = 3$, $kx - y = 2$ has a unique solution, then

   (A) $k = 3$ (B) $k \neq 3$ (C) $k = 4$ (D) $k \neq 4$

2. A system of two linear equations in two variables is inconsistent, if their graphs

   (A) coincide (B) intersect only at a point

   (C) do not intersect at any point (D) cut the x-axis

3. The system of equations $x - 4y = 8$, $3x - 12y = 24$

   (A) has infinitely many solutions (B) has no solution

   (C) has a unique solution (D) may or may not have a solution

4. If one zero of the polynomial $p(x) = (k+4)x^2+13x+3k$ is reciprocal of the other, then k is equal to

(A) 2 (B) 3 (C) 4 (D) 5

5. The sum of two zeros of the polynomial $f(x) = 2x^2 + (p+3)x + 5$ is zero, then the value of p is

(A) 3 (B) 4 (C) –3 (D) –4

6. The remainder when $x^2 - 2x + 7$ is divided by $x+4$ is

(A) 28 (B) 29 (C) 30 (D) 31

7. The quotient when $x^3 - 5x^2 + 7x - 4$ is divided by $x–1$ is

(A) $x^2 + 4x + 3$ (B) $x^2 - 4x + 3$ (C) $x^2 - 4x - 3$ (D) $x^2 + 4x - 3$

8. The GCD of $(x^3 + 1)$ and $x^4 - 1$ is

(A) $x^3 - 1$ (B) $x^3 + 1$ (C) $x + 1$ (D) $x - 1$

9. The GCD of $x^2 - 2xy + y^2$ and $x^4 - y^4$ is

(A) 1 (B) $x+y$ (C) $x–y$ (D) $x^2 - y^2$

10. The LCM of $x^3 - a^3$ and $(x - a)^2$ is

(A) $(x^3 - a^3)(x + a)$ (B) $(x^3 - a^3)(x - a)^2$

(C) $(x - a)^2(x^2 + ax + a^2)$ (D) $(x + a)^2(x^2 + ax + a^2)$

11. The LCM of a^k, a^{k+3}, a^{k+5} where $k \in \mathbb{N}$ is

(A) a^{k+9} (B) a^k (C) a^{k+6} (D) a^{k+5}

12. The lowest form of the rational expression $\dfrac{x^2 + 5x + 6}{x^2 - x - 6}$ is

(A) $\dfrac{x-3}{x+3}$ (B) $\dfrac{x+3}{x-3}$ (C) $\dfrac{x+2}{x-3}$ (D) $\dfrac{x-3}{x+2}$

13. If $\dfrac{a+b}{a-b}$ and $\dfrac{a^3 - b^3}{a^3 + b^3}$ are the two rational expressions, then their product is

(A) $\dfrac{a^2 + ab + b^2}{a^2 - ab + b^2}$ (B) $\dfrac{a^2 - ab + b^2}{a^2 + ab + b^2}$ (C) $\dfrac{a^2 - ab - b^2}{a^2 + ab + b^2}$ (D) $\dfrac{a^2 + ab + b^2}{a^2 - ab - b^2}$

14. On dividing $\dfrac{x^2 - 25}{x+3}$ by $\dfrac{x+5}{x^2 - 9}$ is equal to

(A) $(x-5)(x-3)$ (B) $(x-5)(x+3)$ (C) $(x+5)(x-3)$ (D) $(x+5)(x+3)$

15. If $\dfrac{a^3}{a-b}$ is added with $\dfrac{b^3}{b-a}$, then the new expression is

(A) $a^2 + ab + b^2$ (B) $a^2 - ab + b^2$ (C) $a^3 + b^3$ (D) $a^3 - b^3$

16. The square root of $49(x^2 - 2xy + y^2)^2$ is

(A) $7|x - y|$ (B) $7(x + y)(x - y)$ (C) $7(x + y)^2$ (D) $7(x - y)^2$

17. The square root of $x^2 + y^2 + z^2 - 2xy + 2yz - 2zx$

(A) $|x + y - z|$ (B) $|x - y + z|$ (C) $|x + y + z|$ (D) $|x - y - z|$

18. The square root of $121x^4y^8z^6(l-m)^2$ is

(A) $11x^2y^4z^4|l-m|$ (B) $11x^4y^4|z^3(l-m)|$

(C) $11x^2y^4z^6|l-m|$ (D) $11x^2y^4|z^3(l-m)|$

19. If $ax^2+bx+c=0$ has equal roots, then c is equal

(A) $\frac{b^2}{2a}$ (B) $\frac{b^2}{4a}$ (C) $-\frac{b^2}{2a}$ (D) $-\frac{b^2}{4a}$

20. If $x^2+5kx+16=0$ has no real roots, then

(A) $k>\frac{8}{5}$ (B) $k>-\frac{8}{5}$ (C) $-\frac{8}{5}<k<\frac{8}{5}$ (D) $0<k<\frac{8}{5}$

21. A quadratic equation whose one root is 3 is

(A) $x^2-6x-5=0$ (B) $x^2+6x-5=0$

(C) $x^2-5x-6=0$ (D) $x^2-5x+6=0$

22. The common root of the equations $x^2-bx+c=0$ and $x^2+bx-a=0$ is

(A) $\frac{c+a}{2b}$ (B) $\frac{c-a}{2b}$ (C) $\frac{c+b}{2a}$ (D) $\frac{a+b}{2c}$

23. If α,β are the roots of $ax^2+bx+c=0$ $a\neq 0$, then the wrong statement is

(A) $\alpha^2+\beta^2=\frac{b^2-2ac}{a^2}$ (B) $\alpha\beta=\frac{c}{a}$

(C) $\alpha+\beta=\frac{b}{a}$ (D) $\frac{1}{\alpha}+\frac{1}{\beta}=-\frac{b}{c}$

24. If α and β are the roots of $ax^2+bx+c=0$, then one of the quadratic equations whose roots are $\frac{1}{\alpha}$ and $\frac{1}{\beta}$, is

(A) $ax^2+bx+c=0$ (B) $bx^2+ax+c=0$

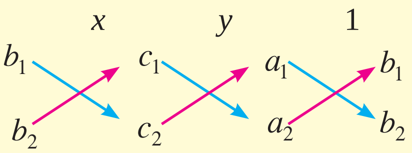
(C) $cx^2+bx+a=0$ (D) $cx^2+ax+b=0$

25. Let $b=a+c$. Then the equation $ax^2+bx+c=0$ has equl roots, if

(A) a = c (B) a = – c (C) a = 2c (D) a = –2c

Points to Remember

- A set of finite number of linear equations in two variables x and y is called a system of linear equations in x and y. Such a system is also called simultaneous equations.
- Eliminating one of the variables first and then solving a system is called method of elimination.
- The following arrow diagram helps us very much to apply the method of cross multiplication in solving $a_1x+b_1y+c_1=0$, $a_2x+b_2y+c_2=0$.



- A real number k is said to be a zero of a polynomial $p(x)$, if $p(k)=0$.

- The basic relationships between zeros and coefficients of a quadratic polynomial $p(x) = ax^2 + bx + c$ are

  Sum of zeros $= -\dfrac{b}{a} = -\dfrac{\text{coefficient of } x}{\text{coefficient of } x^2}$

  Product of zeros $= \dfrac{c}{a} = \dfrac{\text{constant term}}{\text{coefficient of } x^2}$

- (i) For any polynomial $p(x)$, $x = a$ is zero if and only if $p(a) = 0$.

  (ii) $x - a$ is a factor for $p(x)$ if and only if $p(a) = 0$.

- GCD of two or more algebraic expressions is the expression of highest degree which divides each of them without remainder.

- LCM of two or more algebraic expressions is the expression of lowest degree which is divisible by each of them without remainder.

- The product of LCM and GCD of any two polynomials is equal to the product of the two polynomials.

- Let $a \in \mathbb{R}$ be a non negative real number. A square root of a, is a real number b such that $b^2 = a$. The square root of a is denoted by $\sqrt[2]{a}$ or $\sqrt{a}$.

- A quadratic equation in the variable x is of the form $ax^2 + bx + c = 0$, where a,b,c are real numbers and $a \neq 0$.

- A quadratic equation can be solved by (i) the method of factorization (ii) the method of completing square (iii) using a quadratic formula.

- The roots of a quadratic equation $ax^2 + bx + c = 0$ are given by $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, provided $b^2 - 4ac \geq 0$.

- A quadratic equation $ax^2 + bx + c = 0$ has

  (i) two distinct real roots if $b^2 - 4ac > 0$

  (ii) two equal roots if $b^2 - 4ac = 0$, and

  (iii) no real roots if $b^2 - 4ac < 0$

Do you know?

Fermat's last theorem: The equation $x^n + y^n = z^n$ has no integer solution when $n > 2$. Fermat wrote, " I have discovered a truely remarkable proof which this margin is too small to contain ". No one was able to solve this for over 300 years until British mathematician Andrew Wiles solved it in 1994. Interestingly he came to know about this problem in his city library when he was a high school student.`

4

MATRICES

Number, place, and combination - the three intersecting but distinct spheres of thought to which all mathematical ideas admit of being referred - Sylvester

- Introduction
- Formation of Matrices
- Types of Matrices
- Addition, Subtraction and Multiplication of matrices
- Matrix equations

James Joseph Sylvester

(1814-1897)

England

James Joseph Sylvester made fundamental contributions to matrix theory, invariant theory, number theory and combinatorics. He determined all matrices that commute with a given matrix. He introduced many mathematical terms including "discriminant".

In 1880, the Royal Society of London awarded Sylvester the Copley Medal, a highest award for scientific achievement. In 1901, Royal Society of London instituted the Sylvester medal in his memory, to encourage mathematical research.

4.1 Introduction

In this chapter we are going to discuss an important mathematical object called "MATRIX". Here, we shall introduce matrices and study the basics of matrix algebra.

Matrices were formulated and developed as a concept during 18th and 19th centuries. In the beginning, their development was due to transformation of geometric objects and solution of linear equations. However matrices are now one of the most powerful tools in mathematics. Matrices are useful because they enable us to consider an array of many numbers as a single object and perform calculations with these symbols in a very compact form. The " mathematical shorthand" thus obtained is very elegant and powerful and is suitable for various practical problems.

The term "Matrix" for arrangement of numbers, was introduced in 1850 by James Joseph Sylvester. "Matrix" is the Latin word for womb, and it retains that sense in English. It can also mean more generally any place in which something is formed or produced.

Now let us consider the following system of linear equations in x and y :

$$3x - 2y = 4 \qquad (1)$$

$$2x + 5y = 9 \qquad (2)$$

We already know how to get the solution (2, 1) of this system by the method of elimination (also known as Gaussian Elimination method), where only the coefficients are used and not the variables. The same method can easily be executed and the solution can thus be obtained using matrix algebra.

4.2 Formation of matrices

Let us consider some examples of the ways that matrices can arise.

Kumar has 10 pens. We may express it as (10), with the understanding that the number inside ( ) is the number of pens that Kumar has.

Now, if Kumar has 10 pens and 7 pencils, we may express it as (10 7) with the understanding that the first number inside ( ) is the number of pens while the other one is the number of pencils.

Look at the following information :

Pens and Pencils owned by Kumar and his friends Raju and Gopu are as given below.

Kumar has 10 pens and 7 pencils
Raju has 8 pens and 4 pencils
Gopu has 6 pens and 5 pencils

This can be arranged in tabular form as follows:

	Pens	Pencils
Kumar	10	7
Raju	8	4
Gopu	6	5

This can be expressed in a rectangular array where the entries denote the number of respective items.

$$\text{(i)} \quad \begin{array}{cc} \begin{pmatrix} 10 & 7 \\ 8 & 4 \\ 6 & 5 \end{pmatrix} & \begin{array}{l} \leftarrow \text{first row} \\ \leftarrow \text{second row} \\ \leftarrow \text{third row} \end{array} \end{array}$$

$$\begin{array}{cc} \uparrow & \uparrow \\ \text{first} & \text{second} \\ \text{column} & \text{column} \end{array}$$

The same information can also be arranged in tabular form as :

	Kumar	Raju	Gopu
Pens	10	8	6
Pencils	7	4	5

This can be expressed in a rectangular array.

$$\text{(ii)} \quad \begin{array}{cc} \begin{pmatrix} 10 & 8 & 6 \\ 7 & 4 & 5 \end{pmatrix} & \begin{array}{l} \leftarrow \text{first row} \\ \leftarrow \text{second row} \end{array} \end{array}$$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \text{first} & \text{second} & \text{third} \\ \text{column} & \text{column} & \text{column} \end{array}$$

In arrangement (i), the entries in the first column represent the number of pens of Kumar, Raju and Gopu respectively and the second column represents the number of pencils owned by Kumar, Raju and Gopu respectively.

Similarly, in arrangement (ii), the entries in the first row represent the number of pens of Kumar, Raju and Gopu respectively. The entries in the second row represent the number of pencils owned by Kumar, Raju and Gopu respectively.

An arrangement or display of numbers of the above kind is called a MATRIX.

Definition

A matrix is a rectangular array of numbers in rows and columns enclosed within square brackets or parenthesis.

A matrix is usually denoted by a single capital letter like $A, B, X, Y, \cdots$. The numbers that make up a matrix are called entries or elements of the matrix. Each horizontal arrangement in a matrix is called a row of that matrix. Each vertical arrangement in a matrix is called a column of that matrix.

Some examples of matrices are

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \quad B = \begin{bmatrix} 2 & 0 & -1 \\ 3 & -8 & 9 \\ 1 & 5 & -1 \end{bmatrix} \text{ and } C = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

4.2.1 General form of a matrix

A matrix A with m rows and n columns, is of the form

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1j} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2j} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mj} & \ldots & a_{mn} \end{pmatrix}$$

where $a_{11}, a_{12}, a_{13}, \ldots$ are the elements of the matrix. The above matrix can also be written as $A = [a_{ij}]_{m \times n}$ or $A = (a_{ij})_{m \times n}$, where $i = 1, 2, 3, \ldots, m$. and $j = 1, 2, 3, \ldots, n$.

Here, a_{ij} is the element of the matrix lying on the intersection of the i^{th} row and j^{th} column of A.

For example, if $A = \begin{pmatrix} 4 & 5 & 3 \\ 6 & 2 & 1 \\ 7 & 8 & 9 \end{pmatrix}$, then $a_{23} = 1$, the element which occurs in the second row and third column.

Similarly, $a_{11} = 4$, $a_{12} = 5$, $a_{13} = 3$, $a_{21} = 6$, $a_{22} = 2$, $a_{31} = 7$, $a_{32} = 8$ and $a_{33} = 9$.

4.2.2 Order or dimension of a matrix

If a matrix A has m rows and n columns, then we say that the order of A is $m \times n$ (Read as m by n).

The matrix

$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ has 2 rows and 3 columns. So, the order of A is 2×3.

Note

In a $m \times n$ matrix, the first letter m always denotes the number of rows and the second letter n always denotes the number of columns.

4.3 Types of matrices

Let us learn certain types of matrices.

(i) Row matrix

A matrix is said to be a **row matrix** if it has only one row. A row matrix is also called as a row vector.

For example, $A = (5 \ \ 3 \ \ 4 \ \ 1)$ and $B = (-3 \ \ 0 \ \ 5)$ are row matrices of orders 1×4 and 1×3 respectively.

In general, $A = (a_{ij})_{1 \times n}$ is a row matrix of order $1 \times n$.

(ii) Column matrix

A matrix is said to be a column matrix if it has only one column. It is also called as a column vector.

For example, $A = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$ are column matrices of orders 2×1 and 3×1 respectively.

In general, $A = [a_{ij}]_{m \times 1}$ is a column matrix of order $m \times 1$.

(iii) Square matrix

A matrix in which the number of rows and the number of columns are equal is said to be a square matrix**.** For example,

$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 0 & 2 \\ 1 & 5 & -7 \\ 7 & 6 & 1 \end{pmatrix}$ are square matrices of orders 2 and 3 respectively.

In general, $A = [a_{ij}]_{m \times m}$ is a square matrix of order m.

The elements $a_{11}, a_{22}, a_{33}, \cdots, a_{mm}$ are called principal or leading diagonal elements of the square matrix A.

(iv) Diagonal matrix

A square matrix in which all the elements above and below the leading diagonal are equal to zero, is called a diagonal matrix. For example,

$A = \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ are diagonal matrices of orders 2 and 3 respectively. In general, $A = [a_{ij}]_{m \times m}$ is said to be a diagonal matrix if $a_{ij} = 0$ for all $i \neq j$.

Note

Some of the leading diagonal elements of a diagonal matrix may be zero.

(v) Scalar matrix

A diagonal matrix in which all the elements along the leading diagonal are equal to a non-zero constant is called a scalar matrix. For example,

$$A = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} \text{ and } B = \begin{pmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{pmatrix} \text{ are scalar matrices of orders 2 and 3 respectively.}$$

In general, $A = [a_{ij}]_{m\times m}$ is said to be a scalar matrix if $a_{ij} = \begin{cases} 0, \text{when } i \neq j \\ k, \text{when } i = j \end{cases}$ where k is a constant.

(vi) Unit matrix

A diagonal matrix in which all the leading diagonal entries are 1 is called a unit matrix. A unit matrix of order n is denoted by I_n. For example,

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ are unit matrices of orders 2 and 3 respectively.}$$

In general, a square matrix $A = (a_{ij})_{n\times n}$ is a unit matrix if $a_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$

Note

A unit matrix is also called an identity matrix with respect to multiplication. Every unit matrix is clearly a scalar matrix. However a scalar matrix need not be a unit matrix. A unit matrix plays the role of the number 1 in numbers.

(vii) Null matrix or Zero-matrix

A matrix is said to be a null matrix or zero-matrix if each of its elements is zero. It is denoted by O. For example,

$$O = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ are null matrices of order } 2\times 3 \text{ and } 2\times 2.$$

Note

(i) A zero-matrix need not be a square matrix. (ii) Zero-matrix plays the role of the number zero in numbers. (iii) A matrix does not change if the zero-matrix of same order is added to it or subtracted from it.

(viii) Transpose of a matrix

Definition The transpose of a matrix A is obtained by interchanging rows and columns of the matrix A and it is denoted by A^T (read as A transpose). For example,

$$\text{if } A = \begin{pmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{pmatrix}, \text{ then } A^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \\ 5 & 6 \end{pmatrix}$$

In general, if $A = [a_{ij}]_{m\times n}$ then

$$A^T = [b_{ij}]_{n\times m}, \text{ where } b_{ij} = a_{ji}, \text{ for } i = 1, 2, \cdots, n \text{ and } j = 1, 2, \cdots, m.$$

Example 4.1

The table shows a five-day forecast indicating high (H) and low (L) temperatures in Fahrenheit. Organise the temperatures in a matrix where the first and second rows represent the High and Low temperatures respectively and identify which day will be the warmest?

Mon	Tue	Wed	Thu	Fri
H 88	H 90	H 86	H 84	H 85
L 54	L 56	L 53	L 52	L 52

Solution The above information can be represented in matrix form as

$$A = \begin{array}{c} \\ H \\ L \end{array}\begin{array}{c} \begin{array}{ccccc} \text{Mon} & \text{Tue} & \text{Wed} & \text{Thu} & \text{Fri} \end{array} \\ \begin{pmatrix} 88 & 90 & 86 & 84 & 85 \\ 54 & 56 & 53 & 52 & 52 \end{pmatrix} \end{array}. \quad \text{That is , } A = \begin{pmatrix} 88 & 90 & 86 & 84 & 85 \\ 54 & 56 & 53 & 52 & 52 \end{pmatrix}$$

By reading through the first row (High), the warmest day is Tuesday.

Example 4.2

The amount of fat, carbohydrate and protein in grams present in each food item respectively are as follows:

	Item 1	Item 2	Item 3	Item 4
Fat	5	0	1	10
Carbohydrate	0	15	6	9
Protein	7	1	2	8

Use the information to write 3×4 and 4×3 matrices.

Solution The above information can be represented in the form of 3×4 matrix as

$$A = \begin{pmatrix} 5 & 0 & 1 & 10 \\ 0 & 15 & 6 & 9 \\ 7 & 1 & 2 & 8 \end{pmatrix}$$ where the columns correspond to food items. We write

a 4×3 matrix as $B = \begin{pmatrix} 5 & 0 & 7 \\ 0 & 15 & 1 \\ 1 & 6 & 2 \\ 10 & 9 & 8 \end{pmatrix}$ where the rows correspond to food items.

Example 4.3

Let $A = [a_{ij}] = \begin{pmatrix} 1 & 4 & 8 \\ 6 & 2 & 5 \\ 3 & 7 & 0 \\ 9 & -2 & -1 \end{pmatrix}$. Find

(i) the order of the matrix (ii) the elements a_{13} and a_{42} (iii) the position of the element 2.

Solution (i) Since the matrix A has 4 rows and 3 columns, A is of order 4×3.

(ii) The element a_{13} is in the first row and third column. $\therefore\ a_{13} = 8$.
Similarly, $a_{42} = -2$, the element in 4th row and 2nd column.

(iii) The element 2 occurs in 2nd row and 2nd column $\therefore\ a_{22} = 2$.

Example 4.4

Construct a 2×3 matrix $A = [a_{ij}]$ whose elements are given by $a_{ij} = |2i - 3j|$

Solution In general a 2×3 matrix is given by

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$

Now, $a_{ij} = |2i - 3j|$ where $i = 1,2$ and $j = 1,2,3$

$$a_{11} = |2(1) - 3(1)| = |-1| = 1,\ a_{12} = |2(1) - 3(2)| = 4,\ a_{13} = |2(1) - 3(3)| = 7$$

$$a_{21} = |2(2) - 3| = 1,\qquad a_{22} = |2(2) - 3(2)| = 2,\ a_{23} = |2(2) - 9| = 5$$

Hence the required matrix $A = \begin{pmatrix} 1 & 4 & 7 \\ 1 & 2 & 5 \end{pmatrix}$

Example 4.5

If $A = \begin{pmatrix} 8 & 5 & 2 \\ 1 & -3 & 4 \end{pmatrix}$, then find A^T and $(A^T)^T$

Solution

$$A = \begin{pmatrix} 8 & 5 & 2 \\ 1 & -3 & 4 \end{pmatrix}$$

The transpose A^T of a matrix *A*, is obtained by interchanging rows and columns of the matrix *A*.

Thus, $A^T = \begin{pmatrix} 8 & 1 \\ 5 & -3 \\ 2 & 4 \end{pmatrix}$

Similarly $(A^T)^T$ is obtained by interchanging rows and columns of the matrix A^T.

Hence, $(A^T)^T = \begin{pmatrix} 8 & 5 & 2 \\ 1 & -3 & 4 \end{pmatrix}$

Note

From the above example, we see that $(A^T)^T = A$. In fact, it is true that $(B^T)^T = B$ for any matrix *B*. Also, $(kA)^T = kA^T$ for any scalar *k*.

Exercise 4.1

1. The rates for the entrance tickets at a water theme park are listed below:

	Week Days rates(₹)	Week End rates(₹)
Adult	400	500
Children	200	250
Senior Citizen	300	400

Write down the matrices for the rates of entrance tickets for adults, children and senior citizens. Also find the dimensions of the matrices.

2. There are 6 Higher Secondary Schools, 8 High Schools and 13 Primary Schools in a town. Represent these data in the form of 3×1 and 1×3 matrices.

3. Find the order of the following matrices.

(i) $\begin{pmatrix} 1 & -1 & 5 \\ -2 & 3 & 4 \end{pmatrix}$ (ii) $\begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$ (iii) $\begin{pmatrix} 3 & -2 & 6 \\ 6 & -1 & 1 \\ 2 & 4 & 5 \end{pmatrix}$ (iv) $\begin{pmatrix} 3 & 4 & 5 \end{pmatrix}$ (v) $\begin{pmatrix} 1 & 2 \\ -2 & 3 \\ 9 & 7 \\ 6 & 4 \end{pmatrix}$

4. A matrix has 8 elements. What are the possible orders it can have?

5. A matrix consists of 30 elements. What are the possible orders it can have?.

6. Construct a 2×2 matrix $A = [a_{ij}]$ whose elements are given by

(i) $a_{ij} = ij$ (ii) $a_{ij} = 2i - j$ (iii) $a_{ij} = \dfrac{i-j}{i+j}$

7. Construct a 3×2 matrix $A = [a_{ij}]$ whose elements are given by

(i) $a_{ij} = \dfrac{i}{j}$ (ii) $a_{ij} = \dfrac{(i-2j)^2}{2}$ (iii) $a_{ij} = \dfrac{|2i-3j|}{2}$

8. If $A = \begin{pmatrix} 1 & -1 & 3 & 2 \\ 5 & -4 & 7 & 4 \\ 6 & 0 & 9 & 8 \end{pmatrix}$, (i) find the order of the matrix (ii) write down the elements a_{24} and a_{32} (iii) in which row and column does the element 7 occur?

9. If $A = \begin{pmatrix} 2 & 3 \\ 4 & 1 \\ 5 & 0 \end{pmatrix}$, then find the transpose of A.

10. If $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & -5 \\ 3 & -5 & 6 \end{pmatrix}$, then verify that $(A^T)^T = A$.

4.4 Operation on matrices

In this section, we shall discuss the equality of matrices, multiplication of a matrix by a scalar, addition, subtraction and multiplication of matrices.

(i) Equality of matrices

Two matrices $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ are said to be equal if

(i) they are of the same order and

(ii) each element of A is equal to the corresponding element of B, that is $a_{ij} = b_{ij}$ for all i and j.

For example, the matrices $\begin{pmatrix} 6 & 3 \\ 0 & 9 \\ 1 & 5 \end{pmatrix}$ and $\begin{pmatrix} 6 & 0 & 1 \\ 3 & 9 & 5 \end{pmatrix}$ are not equal as the orders of the matrices are different.

Also $\begin{pmatrix} 1 & 2 \\ 8 & 5 \end{pmatrix} \neq \begin{pmatrix} 1 & 8 \\ 2 & 5 \end{pmatrix}$, since some of the corresponding elements are not equal.

Example 4.6

Find the values of x, y and z if $\begin{pmatrix} x & 5 & 4 \\ 5 & 9 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 5 & z \\ 5 & y & 1 \end{pmatrix}$

Solution As the given matrices are equal, their corresponding elements must be equal. Comparing the corresponding elements, we get $x = 3$, $y = 9$ and $z = 4$.

Example 4.7

Solve : $\begin{pmatrix} y \\ 3x \end{pmatrix} = \begin{pmatrix} 6 - 2x \\ 31 + 4y \end{pmatrix}$

Solution Since the matrices are equal, the corresponding elements are equal.

Comparing the corresponding elements, we get $y = 6 - 2x$ and $3x = 31 + 4y$.

Using $y = 6 - 2x$ in the other equation, we get $3x = 31 + 4(6 - 2x)$

$$3x = 31 + 24 - 8x$$

$\therefore$ $x = 5$ and hence $y = 6 - 2(5) = -4$.

Thus, $x = 5$ and $y = -4$.

(ii) Multiplication of a matrix by a scalar

Definition

For a given matrix $A = [a_{ij}]_{m\times n}$ and a scalar (real number) k, we define a new matrix $B = [b_{ij}]_{m\times n}$, where $b_{ij} = ka_{ij}$ for all i and j.

Thus, the matrix B is obtained by multiplying each entry of A by the scalar k and written as $B = kA$. This multiplication is called scalar multiplication.

For example, if $A = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$ then $kA = k\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} = \begin{pmatrix} ka & kb & kc \\ kd & ke & kf \end{pmatrix}$

Example 4.8

If $A = \begin{pmatrix} -1 & 2 & 4 \\ 3 & 6 & -5 \end{pmatrix}$ then find $3A$

Solution The matrix $3A$ is obtained by multiplying every element of A by 3.

$$3A = 3\begin{pmatrix} -1 & 2 & 4 \\ 3 & 6 & -5 \end{pmatrix} = \begin{pmatrix} 3(-1) & 3(2) & 3(4) \\ 3(3) & 3(6) & 3(-5) \end{pmatrix} = \begin{pmatrix} -3 & 6 & 12 \\ 9 & 18 & -15 \end{pmatrix}$$

(iii) Addition of matrices

Matrices A and B given below show the marks obtained by 3 boys and 3 girls in the subjects Mathematics and Science respectively.

$$\begin{array}{cc} & \text{Mathematics} \\ A = & \begin{pmatrix} 45 & 72 & 81 \\ 30 & 90 & 65 \end{pmatrix} \end{array}\begin{array}{l} \\ \text{Boys} \\ \text{Girls} \end{array} \qquad \begin{array}{cc} & \text{Science} \\ B = & \begin{pmatrix} 51 & 80 & 90 \\ 42 & 85 & 70 \end{pmatrix} \end{array}\begin{array}{l} \\ \text{Boys} \\ \text{Girls} \end{array}$$

To find the total marks obtained by each student, we shall add the corresponding entries of A and B. We write

$$A+B=\begin{pmatrix} 45 & 72 & 81 \\ 30 & 90 & 65 \end{pmatrix}+\begin{pmatrix} 51 & 80 & 90 \\ 42 & 85 & 70 \end{pmatrix}$$

$$=\begin{pmatrix} 45+51 & 72+80 & 81+90 \\ 30+42 & 90+85 & 65+70 \end{pmatrix}=\begin{pmatrix} 96 & 152 & 171 \\ 72 & 175 & 135 \end{pmatrix}$$

The final matrix shows that the first boy scores a total of 96 marks in Mathematics and Science. Similarly, the last girl scores a total of 135 marks in Mathematics and Science.

Hence, we observe that the sum of two matrices of same order is a matrix obtained by adding the corresponding entries of the given matrices.

Definition

If $A=[a_{ij}]_{m\times n}$ and $B=[b_{ij}]_{m\times n}$ are two matrices of the same order, then the addition of A and B is a matrix $C=[c_{ij}]_{mxn}$, where $c_{ij}=a_{ij}+b_{ij}$ for all i and j.

Note that the operation of addition on matrices is defined as for numbers. The addition of two matrices A and B is denoted by $A+B$. Addition is not defined for matrices of different orders.

Example 4.9

Let $A=\begin{pmatrix} 8 & 3 & 2 \\ 5 & 9 & 1 \end{pmatrix}$ and $B=\begin{pmatrix} 1 & -1 \\ 3 & 0 \end{pmatrix}$. Find $A+B$ if it exists.

Solution Since A is order of 2×3 and B is of order 2×2, addition of matrices A and B is not possible.

Example 4.10

If $A=\begin{pmatrix} 5 & 6 & -2 & 3 \\ 1 & 0 & 4 & 2 \end{pmatrix}$ and $B=\begin{pmatrix} 3 & -1 & 4 & 7 \\ 2 & 8 & 2 & 3 \end{pmatrix}$, then find $A+B$.

Solution Since A and B are of the same order 2×4, addition of A and B is defined.

So, $$A+B=\begin{pmatrix} 5 & 6 & -2 & 3 \\ 1 & 0 & 4 & 2 \end{pmatrix}+\begin{pmatrix} 3 & -1 & 4 & 7 \\ 2 & 8 & 2 & 3 \end{pmatrix}$$

$$=\begin{pmatrix} 5+3 & 6-1 & -2+4 & 3+7 \\ 1+2 & 0+8 & 4+2 & 2+3 \end{pmatrix}$$

Thus, $$A+B=\begin{pmatrix} 8 & 5 & 2 & 10 \\ 3 & 8 & 6 & 5 \end{pmatrix}$$

(iv) Negative of a matrix

The negative of a matrix $A=[a_{ij}]_{mxn}$ is denoted by $-A$ and is defined as $-A=(-1)A$. That is, $-A=[b_{ij}]_{mxn}$, where $b_{ij}=-a_{ij}$ for all i and j.

(v) Subtraction of matrices

If $A=[a_{ij}]_{m\times n}$ and $B=[b_{ij}]_{m\times n}$ are two matrices of the same order, then the subtraction $A-B$ is defined as $A-B=A+(-1)B$. That is, $A-B=[c_{ij}]$ where $c_{ij}=a_{ij}-b_{ij}$ for all i and j.

Example 4.11

Matrix A shows the weight of four boys and four girls in kg at the beginning of a diet programme to lose weight. Matrix B shows the corresponding weights after the diet programme.

$$A = \begin{pmatrix} 35 & 40 & 28 & 45 \\ 42 & 38 & 41 & 30 \end{pmatrix}\begin{array}{l}\text{Boys}\\ \text{Girls}\end{array}, \quad B = \begin{pmatrix} 32 & 35 & 27 & 41 \\ 40 & 30 & 34 & 27 \end{pmatrix}\begin{array}{l}\text{Boys}\\ \text{Girls}\end{array}$$

Find the weight loss of the Boys and Girls.

Solution Weight loss matrix $A - B = \begin{pmatrix} 35 & 40 & 28 & 45 \\ 42 & 38 & 41 & 30 \end{pmatrix} - \begin{pmatrix} 32 & 35 & 27 & 41 \\ 40 & 30 & 34 & 27 \end{pmatrix}$

$$= \begin{pmatrix} 3 & 5 & 1 & 4 \\ 2 & 8 & 7 & 3 \end{pmatrix}.$$

4.5 Properties of matrix addition

(i) Matrix addition is commutative

If A and B are any two matrices of same order, then $A+B = B+A$

(ii) Matrix addition is associative

If A, B and C are any three matrices of same order, then $A + (B + C) = (A + B) + C$

(iii) Existence of additive identity

Null or zero matrix is the additive identity for matrix addition. If A is a matrix of order $m \times n$, then $A + O = O + A = A$, where O is the null matrix of order $m \times n$,

(iv) Existence of additive inverse

For a matrix A, B is called the additive inverse of A if $B + A = A + B = O$.

Since $A + (-A) = (-A) + A = O$, $-A$ is the additive inverse of A.

Note

The additive inverse of a matrix is its negative matrix and it is unique (only one).

Exercise 4.2

1. Find the values of x, y and z from the matrix equation

$$\begin{pmatrix} 5x + 2 & y - 4 \\ 0 & 4z + 6 \end{pmatrix} = \begin{pmatrix} 12 & -8 \\ 0 & 2 \end{pmatrix}$$

2. Solve for x and y if $\begin{pmatrix} 2x + y \\ x - 3y \end{pmatrix} = \begin{pmatrix} 5 \\ 13 \end{pmatrix}$

3. If $A = \begin{pmatrix} 2 & 3 \\ -9 & 5 \end{pmatrix} - \begin{pmatrix} 1 & 5 \\ 7 & -1 \end{pmatrix}$, then find the additive inverse of A.

4. Let $A = \begin{pmatrix} 3 & 2 \\ 5 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 8 & -1 \\ 4 & 3 \end{pmatrix}$. Find the matrix C if $C = 2A + B$.

5. If $A = \begin{pmatrix} 4 & -2 \\ 5 & -9 \end{pmatrix}$ and $B = \begin{pmatrix} 8 & 2 \\ -1 & -3 \end{pmatrix}$ find $6A - 3B$.

6. Find a and b if $a\begin{pmatrix} 2 \\ 3 \end{pmatrix} + b\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 \\ 5 \end{pmatrix}$.

7. Find X and Y if $2X + 3Y = \begin{pmatrix} 2 & 3 \\ 4 & 0 \end{pmatrix}$ and $3X + 2Y = \begin{pmatrix} 2 & -2 \\ -1 & 5 \end{pmatrix}$.

8. Solve for x and y if $\begin{pmatrix} x^2 \\ y^2 \end{pmatrix} + 3\begin{pmatrix} 2x \\ -y \end{pmatrix} = \begin{pmatrix} -9 \\ 4 \end{pmatrix}$.

9. If $A = \begin{pmatrix} 3 & 2 \\ 5 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -2 \\ 2 & 3 \end{pmatrix}$ and $O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ then
verify: (i) $A + B = B + A$ (ii) $A + (-A) = O = (-A) + A$.

10. If $A = \begin{pmatrix} 4 & 1 & 2 \\ 1 & -2 & 3 \\ 0 & 3 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 0 & 4 \\ 6 & 2 & 8 \\ 2 & 4 & 6 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{pmatrix}$, then
verify that $A + (B + C) = (A + B) + C$.

11. An electronic company records each type of entertainment device sold at three of their branch stores so that they can monitor their purchases of supplies. The sales in two weeks are shown in the following spreadsheets.

		T.V.	DVD	Videogames	CD Players
Week I	Store I	30	15	12	10
	Store II	40	20	15	15
	Store III	25	18	10	12
Week II	Store I	25	12	8	6
	Store II	32	10	10	12
	Store III	22	15	8	10

Find the sum of the items sold out in two weeks using matrix addition.

12. The fees structure for one-day admission to a swimming pool is as follows:

Daily Admission Fees in ₹		
Member	Children	Adult
Before 2.00 p.m.	20	30
After 2.00 p.m.	30	40
Non-Member		
Before 2.00 p.m.	25	35
After 2.00 p.m.	40	50

Write the matrix that represents the additional cost for non-membership.

4.6 Multiplication of matrices

Suppose that Selvi wants to buy 3 pens and 2 pencils, while Meena needs 4 pens and 5 pencils. Each pen and pencil cost ₹10 and ₹5 respectively. How much money does each need to spend?

Clearly, Since $3\times 10+2\times 5=40$, Selvi needs ₹ 40.

Since $4\times 10+5\times 5=65$, Meena needs ₹ 65.

We can also do this using matrix multiplication.

Let us write the above information as follows:

$$\begin{array}{ccc} \text{Requirements} & \text{Price (in ₹)} & \text{Money Needed (in ₹)} \\ \begin{array}{c}\text{Selvi}\\ \text{Meena}\end{array}\begin{pmatrix}3 & 2\\ 4 & 5\end{pmatrix} & \begin{pmatrix}10\\ 5\end{pmatrix} & \begin{pmatrix}3\times 10+2\times 5\\ 4\times 10+5\times 5\end{pmatrix}=\begin{pmatrix}40\\ 65\end{pmatrix}\end{array}$$

Suppose the cost of each pen and pencil in another shop are ₹8 and ₹4 respectively. The money required by Selvi and Meena will be $3\times 8+2\times 4$ = ₹32 and $4\times 8+5\times 4=$ ₹ 52 . The above information can be represented as

$$\begin{array}{ccc} \text{Requirements} & \text{Price (in ₹)} & \text{Money Needed (in ₹)} \\ \begin{array}{c}\text{Selvi}\\ \text{Meena}\end{array}\begin{pmatrix}3 & 2\\ 4 & 5\end{pmatrix} & \begin{pmatrix}8\\ 4\end{pmatrix} & \begin{pmatrix}3\times 8+2\times 4\\ 4\times 8+5\times 4\end{pmatrix}=\begin{pmatrix}32\\ 52\end{pmatrix}\end{array}$$

Now, the above information in both the cases can be combined in matrix form as shown below.

$$\begin{array}{ccc} \text{Requirements} & \text{Price (in ₹)} & \text{Money needed (in ₹)} \\ \begin{array}{c}\text{Selvi}\\ \text{Meena}\end{array}\begin{pmatrix}3 & 2\\ 4 & 5\end{pmatrix} & \begin{pmatrix}10 & 8\\ 5 & 4\end{pmatrix} & \begin{pmatrix}3\times 10+2\times 5 & 3\times 8+2\times 4\\ 4\times 10+5\times 5 & 4\times 8+5\times 4\end{pmatrix}=\begin{pmatrix}40 & 32\\ 65 & 52\end{pmatrix}\end{array}$$

From the above example, we observe that multiplication of two matrices is possible if the number of columns in the first matrix is equal to the number of rows in the second matrix. Further, for getting the elements of the product matrix, we take rows of the first matrix and columns of the second matrix, multiply them element-wise and sum it.

The following simple example illustrates how to get the elements of the product matrix when the product is defined.

Let $A=\begin{pmatrix}2 & -1\\ 3 & 4\end{pmatrix}$ and $B=\begin{pmatrix}3 & -9\\ 5 & 7\end{pmatrix}$. Then the product of AB is defined and is given by

$$AB=\begin{pmatrix}2 & -1\\ 3 & 4\end{pmatrix}\begin{pmatrix}3 & -9\\ 5 & 7\end{pmatrix}$$

Step 1 : Multiply the numbers in the first row of A by the numbers in the first column of B, add the products, and put the result in the first row and first column of AB.

$$\begin{pmatrix}2 & -1\\ 3 & 4\end{pmatrix}\begin{pmatrix}3 & -9\\ 5 & 7\end{pmatrix}=\begin{pmatrix}2(3)+(-1)5 & \\ & \end{pmatrix}$$

Step 2: Follow the same procedure as in step 1, using the first row of A and second column of B. Write the result in the first row and second column of AB.

$$\begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 3 & -9 \\ 5 & 7 \end{pmatrix} = \begin{pmatrix} 2(3)+(-1)5 & 2(-9)+(-1)7 \\ & \end{pmatrix}$$

Step 3: Follow the same procedure with the second row of A and first column of B. Write the result in the second row and first column of AB.

$$\begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 3 & -9 \\ 5 & 7 \end{pmatrix} = \begin{pmatrix} 2(3)+(-1)5 & 2(-9)+(-1)7 \\ 3(3)+4(5) & \end{pmatrix}$$

Step 4: The procedure is the same for the numbers in the second row of A and second column of B.

$$\begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 3 & -9 \\ 5 & 7 \end{pmatrix} = \begin{pmatrix} 2(3)+(-1)5 & 2(-9)+(-1)7 \\ 3(3)+4(5) & 3(-9)+4(7) \end{pmatrix}$$

Step 5: Simplify to get the product matrix AB

$$\begin{pmatrix} 2(3)+(-1)5 & 2(-9)+(-1)7 \\ 3(3)+4(5) & 3(-9)+4(7) \end{pmatrix} = \begin{pmatrix} 1 & -25 \\ 29 & 1 \end{pmatrix}$$

Definition

If $A=[a_{ij}]_{m\times n}$ and $B=[b_{ij}]_{n\times p}$ then the product matrix AB is defined and is of order $m\times p$. This fact is explained in the following diagram.

$$A_{m\times \boxed{n} \qquad B_{\boxed{n}\times p}}$$

same

product matrix AB is of order $m\times p$

Example 4.12

Determine whether each matrix product is defined or not. If the product is defined, state the dimension of the product matrix.

(i) $A_{2\times 5}$ and $B_{5\times 4}$ (ii) $A_{1\times 3}$ and $B_{4\times 3}$

Solution

(i) Now, the number of columns in A and the number of rows in B are equal.

So, the product AB is defined.

Also, the product matrix AB is of order 2×4.

(ii) Given that A is of order 1×3 and B is of order 4×3

Now, the number of columns in A and the number of rows in B are not equal.

So, the matrix product AB is not defined.

Example 4.13

Solve $\begin{pmatrix}3 & 2\\4 & 5\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}8\\13\end{pmatrix}$

Solution Given that $\begin{pmatrix}3 & 2\\4 & 5\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}8\\13\end{pmatrix}$

$$\Longrightarrow \quad \begin{pmatrix}3x+2y\\4x+5y\end{pmatrix}=\begin{pmatrix}8\\13\end{pmatrix}$$

Equating the corresponding elements, we get

$$3x+2y=8 \quad \text{and} \quad 4x+5y=13$$

$$\Longrightarrow 3x+2y-8=0 \quad \text{and} \quad 4x+5y-13=0.$$

Solving the equations by the method of cross multiplication, we get

$$\begin{array}{cccccc} & x & & y & 1 & \\ 2 & -8 & & 3 & 2 & \\ 5 & -13 & & 4 & 5 & \end{array}$$

$$\Longrightarrow \frac{x}{-26+40}=\frac{y}{-32+39}=\frac{1}{15-8} \quad \Longrightarrow \quad \frac{x}{14}=\frac{y}{7}=\frac{1}{7}$$

Thus, $x=2,\ y=1$

Example 4.14

If $A=\begin{pmatrix}a & b\\c & d\end{pmatrix}$ and $I_2=\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$, then show that $A^2-(a+d)A=(bc-ad)I_2$.

Solution Consider $A^2=A\times A$

$$=\begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}a & b\\c & d\end{pmatrix}=\begin{pmatrix}a^2+bc & ab+bd\\ac+cd & bc+d^2\end{pmatrix} \qquad (1)$$

Now, $(a+d)A=(a+d)\begin{pmatrix}a & b\\c & d\end{pmatrix}$

$$=\begin{pmatrix}a^2+ad & ab+bd\\ac+cd & ad+d^2\end{pmatrix} \qquad (2)$$

From (1) and (2) we get,

$$A^2-(a+d)A=\begin{pmatrix}a^2+bc & ab+bd\\ac+cd & bc+d^2\end{pmatrix}-\begin{pmatrix}a^2+ad & ab+bd\\ac+cd & ad+d^2\end{pmatrix}$$

$$=\begin{pmatrix}bc-ad & 0\\0 & bc-ad\end{pmatrix}=(bc-ad)\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$$

Thus, $A^2-(a+d)A=(bc-ad)I_2$.

4.7 Properties of matrix multiplication

The matrix multiplication does not retain some important properties enjoyed by multiplication of numbers. Some of such properties are (i) $AB\neq BA$ (in general) (ii) $AB=0$ does not imply that either A or B is a zero-matrix and (iii) $AB=AC$, A is a non-zero matrix, does not imply always that $B=C$.

For example, let $A=\begin{pmatrix}0&0\\0&1\end{pmatrix}$, $B=\begin{pmatrix}1&2\\3&4\end{pmatrix}$, $C=\begin{pmatrix}5&6\\3&4\end{pmatrix}$ and $D=\begin{pmatrix}1&0\\0&0\end{pmatrix}$. Then,

(i) $AB \neq BA$ (ii) $AD = O$, however, A and D are not zero-matrices and (iii) $AB = AC$, but $B \neq C$. Let us see some properties of matrix multiplication through examples.

(i) Matrix multiplication is not commutative in general

If A and B are two matrices and if AB and BA both are defined, it is not necessary that $AB = BA$.

Example 4.15

If $A=\begin{pmatrix}8&-7\\-2&4\\0&3\end{pmatrix}$ and $B=\begin{pmatrix}9&-3&2\\6&-1&-5\end{pmatrix}$, then find AB and BA if they exist.

Solution The matrix A is of order 3×2 and B is of order 2×3. Thus, both the products AB and BA are defined.

$$\text{Now, } AB=\begin{pmatrix}8&-7\\-2&4\\0&3\end{pmatrix}\begin{pmatrix}9&-3&2\\6&-1&-5\end{pmatrix}$$

$$=\begin{pmatrix}72-42&-24+7&16+35\\-18+24&6-4&-4-20\\0+18&0-3&0-15\end{pmatrix}=\begin{pmatrix}30&-17&51\\6&2&-24\\18&-3&-15\end{pmatrix}$$

Similarly,

$$BA=\begin{pmatrix}9&-3&2\\6&-1&-5\end{pmatrix}\begin{pmatrix}8&-7\\-2&4\\0&3\end{pmatrix}=\begin{pmatrix}78&-69\\50&-61\end{pmatrix}. \qquad \text{(Note that } AB\neq BA\text{)}$$

Remarks

Multiplication of two diagonal matrices of same order is commutative. Also, under matrix multiplication unit matrix commutes with any square matrix of same order.

(ii) Matrix multiplication is always associative

For any three matrices A, B and C, we have $(AB)C = A(BC)$, whenever both sides of the equality are defined.

(iii) Matrix multiplication is distributive over addition

For any three matrices A, B and C, we have (i) $A(B+C)=AB+AC$

(ii) $(A+B)C=AC+BC$, whenever both sides of equality are defined.

Example 4.16

If $A=\begin{pmatrix}3&2\\-1&4\end{pmatrix}$, $B=\begin{pmatrix}-2&5\\6&7\end{pmatrix}$ and $C=\begin{pmatrix}1&1\\-5&3\end{pmatrix}$ verify that $A(B+C)=AB+AC$

Solution Now, $B+C=\begin{pmatrix}-2&5\\6&7\end{pmatrix}+\begin{pmatrix}1&1\\-5&3\end{pmatrix}=\begin{pmatrix}-1&6\\1&10\end{pmatrix}$

$$\text{Thus, } A(B+C)=\begin{pmatrix}3&2\\-1&4\end{pmatrix}\begin{pmatrix}-1&6\\1&10\end{pmatrix}=\begin{pmatrix}-1&38\\5&34\end{pmatrix} \qquad (1)$$

Now, $AB+AC=\begin{pmatrix}3&2\\-1&4\end{pmatrix}\begin{pmatrix}-2&5\\6&7\end{pmatrix}+\begin{pmatrix}3&2\\-1&4\end{pmatrix}\begin{pmatrix}1&1\\-5&3\end{pmatrix}$

$$=\begin{pmatrix}-6+12&15+14\\2+24&-5+28\end{pmatrix}+\begin{pmatrix}3-10&3+6\\-1-20&-1+12\end{pmatrix}$$

$$=\begin{pmatrix}6&29\\26&23\end{pmatrix}+\begin{pmatrix}-7&9\\-21&11\end{pmatrix}$$

$$=\begin{pmatrix}-1&38\\5&34\end{pmatrix} \qquad (2)$$

From (1) and (2), we have $A(B+C)=AB+AC$.

(iv) Existence of multiplicative identity

In ordinary algebra we have the number 1, which has the property that its product with any number is the number itself. We now introduce an analogous concept in matrix algebra.

For any square matrix A of order n, we have $AI=IA=A$, where I is the unit matrix of order n. Hence, I is known as the identity matrix under multiplication.

Example 4.17

If $A=\begin{pmatrix}1&3\\9&-6\end{pmatrix}$, then verify $AI=IA=A$, where I is the unit matrix of order 2.

Solution

Now, $AI=\begin{pmatrix}1&3\\9&-6\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix}=\begin{pmatrix}1+0&0+3\\9+0&0-6\end{pmatrix}=\begin{pmatrix}1&3\\9&-6\end{pmatrix}=A$

Also, $IA=\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}1&3\\9&-6\end{pmatrix}=\begin{pmatrix}1+0&3+0\\0+9&0-6\end{pmatrix}=\begin{pmatrix}1&3\\9&-6\end{pmatrix}=A$

Hence $AI=IA=A$.

(v) Existence of multiplicative inverse

If A is a square matrix of order n, and if there exists a square matrix B of the same order n, such that $AB=BA=I$, where I is the unit matrix of order n, then B is called the multiplicative inverse matrix of A and it is denoted by A^{-1}.

Note

(i) Some of the square matrices like $\begin{pmatrix}2&3\\4&6\end{pmatrix}$ do not have multiplicative inverses.

(ii) If B is the multiplicative inverse of A, then A is the multiplicative inverse of B.

(iii) If multiplicative inverse of a square matrix exists, then it is unique.

Example 4.18

Prove that $\begin{pmatrix}3&5\\1&2\end{pmatrix}$ and $\begin{pmatrix}2&-5\\-1&3\end{pmatrix}$ are multiplicative inverses to each other.

Solution Now, $\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 6-5 & -15+15 \\ 2-2 & -5+6 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

Also, $\begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 6-5 & 10-10 \\ -3+3 & -5+6 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

$\therefore$ The given matrices are inverses to each other under matrix multiplication.

(vi) Reversal law for transpose of matrices

If A and B are two matrices and if AB is defined , then $(AB)^T = B^T A^T$.

Example 4.19

If $A = \begin{pmatrix} -2 \\ 4 \\ 5 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 3 & -6 \end{pmatrix}$, then verify that $(AB)^T = B^T A^T$.

Solution Now, $AB = \begin{pmatrix} -2 \\ 4 \\ 5 \end{pmatrix}\begin{pmatrix} 1 & 3 & -6 \end{pmatrix} = \begin{pmatrix} -2 & -6 & 12 \\ 4 & 12 & -24 \\ 5 & 15 & -30 \end{pmatrix}$

Thus, $(AB)^T = \begin{pmatrix} -2 & 4 & 5 \\ -6 & 12 & 15 \\ 12 & -24 & -30 \end{pmatrix}$ (1)

Now, $B^T A^T = \begin{pmatrix} 1 \\ 3 \\ -6 \end{pmatrix}\begin{pmatrix} -2 & 4 & 5 \end{pmatrix}$

$= \begin{pmatrix} -2 & 4 & 5 \\ -6 & 12 & 15 \\ 12 & -24 & -30 \end{pmatrix}$ (2)

From (1) and (2), we get $(AB)^T = B^T A^T$.

Exercise 4.3

1. Determine whether the product of the matrices is defined in each case. If so, state the order of the product.

   (i) AB, where $A = [a_{ij}]_{4\times3}$, $B = [b_{ij}]_{3\times2}$ (ii) PQ, where $P = [p_{ij}]_{4\times3}$, $Q = [q_{ij}]_{4\times3}$

   (iii) MN, where $M = [m_{ij}]_{3\times1}$, $N = [n_{ij}]_{1\times5}$ (iv) RS, where $R = [r_{ij}]_{2\times2}$, $S = [s_{ij}]_{2\times2}$

2. Find the product of the matrices, if exists,

   (i) $\begin{pmatrix} 2 & -1 \end{pmatrix}\begin{pmatrix} 5 \\ 4 \end{pmatrix}$ (ii) $\begin{pmatrix} 3 & -2 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ 2 & 7 \end{pmatrix}$

   (iii) $\begin{pmatrix} 2 & 9 & -3 \\ 4 & -1 & 0 \end{pmatrix}\begin{pmatrix} 4 & 2 \\ -6 & 7 \\ -2 & 1 \end{pmatrix}$ (iv) $\begin{pmatrix} 6 \\ -3 \end{pmatrix}\begin{pmatrix} 2 & -7 \end{pmatrix}$

3. A fruit vendor sells fruits from his shop. Selling prices of Apple, Mango and Orange are ₹ 20, ₹ 10 and ₹ 5 each respectively. The sales in three days are given below

Day	Apples	Mangoes	Oranges
1	50	60	30
2	40	70	20
3	60	40	10

Write the matrix indicating the total amount collected on each day and hence find the total amount collected from selling of all three fruits combined.

4. Find the values of x and y if $\begin{pmatrix} 1 & 2 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \begin{pmatrix} x & 0 \\ 9 & 0 \end{pmatrix}$.

5. If $A = \begin{pmatrix} 5 & 3 \\ 7 & 5 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and $C = \begin{pmatrix} -5 \\ -11 \end{pmatrix}$ and if $AX = C$, then find the values of x and y.

6. If $A = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$ then show that $A^2 - 4A + 5I_2 = O$.

7. If $A = \begin{pmatrix} 3 & 2 \\ 4 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 0 \\ 3 & 2 \end{pmatrix}$ then find AB and BA. Are they equal?

8. If $A = \begin{pmatrix} -1 & 2 & 1 \\ 1 & 2 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ and $C = \begin{pmatrix} 2 & 1 \end{pmatrix}$ verify $(AB)C = A(BC)$.

9. If $A = \begin{pmatrix} 5 & 2 \\ 7 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$ verify that $(AB)^T = B^T A^T$.

10. Prove that $A = \begin{pmatrix} 5 & 2 \\ 7 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & -2 \\ -7 & 5 \end{pmatrix}$ are inverses to each other under matrix multiplication.

11. Solve $\begin{pmatrix} x & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -2 & -3 \end{pmatrix}\begin{pmatrix} x \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \end{pmatrix}$.

12. If $A = \begin{pmatrix} 1 & -4 \\ -2 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 6 \\ 3 & -2 \end{pmatrix}$, then prove that $(A+B)^2 \neq A^2 + 2AB + B^2$.

13. If $A = \begin{pmatrix} 3 & 3 \\ 7 & 6 \end{pmatrix}$, $B = \begin{pmatrix} 8 & 7 \\ 0 & 9 \end{pmatrix}$ and $C = \begin{pmatrix} 2 & -3 \\ 4 & 6 \end{pmatrix}$, find $(A+B)C$ and $AC + BC$.

Is $(A+B)C = AC + BC$?

Exercise 4.4

Choose the correct answer.

1. Which one of the following statements is not true?
   (A) A scalar matrix is a square matrix
   (B) A diagonal matrix is a square matrix
   (C) A scalar matrix is a diagonal matrix
   (D) A diagonal matrix is a scalar matrix.

2. Matrix $A = [a_{ij}]_{m\times n}$ is a square matrix if
   (A) $m < n$ (B) $m > n$ (C) $m = 1$ (D) $m = n$

3. If $\begin{pmatrix} 3x+7 & 5 \\ y+1 & 2-3x \end{pmatrix} = \begin{pmatrix} 1 & y-2 \\ 8 & 8 \end{pmatrix}$ then the values of x and y respectively are
   (A) $-2\,,7$ (B) $-\frac{1}{3}, 7$ (C) $-\frac{1}{3}\,, -\frac{2}{3}$ (D) $2\,,-7$

4. If $A = \begin{pmatrix} 1 & -2 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix}$ then $A + B$
   (A) $(0\ \ 0\ \ 0)$ (B) $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$
   (C) (-14) (D) not defined

5. If a matrix is of order 2×3, then the number of elements in the matrix is
   (A) 5 (B) 6 (C) 2 (D) 3

6. If $\begin{pmatrix} 8 & 4 \\ x & 8 \end{pmatrix} = 4\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ then the value of x is
   (A) 1 (B) 2 (C) $\frac{1}{4}$ (D) 4

7. If A is of order 3×4 and B is of order 4×3, then the order of BA is
   (A) 3×3 (B) 4×4 (C) 4×3 (D) not defined

8. If $A\times\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \end{pmatrix}$, then the order of A is
   (A) 2×1 (B) 2×2 (C) 1×2 (D) 3×2

9. If A and B are square matrices such that $AB = I$ and $BA = I$, then B is
   (A) Unit matrix (B) Null matrix
   (C) Multiplicative inverse matrix of A (D) $-A$

10. If $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$, then the values of x and y respectively, are
    (A) $2\,,0$ (B) $0\,,2$ (C) $0\,,-2$ (D) $1\,,1$

11. If $A = \begin{pmatrix} 1 & -2 \\ -3 & 4 \end{pmatrix}$ and $A + B = O$, then B is

(A) $\begin{pmatrix} 1 & -2 \\ -3 & 4 \end{pmatrix}$ (B) $\begin{pmatrix} -1 & 2 \\ 3 & -4 \end{pmatrix}$ (C) $\begin{pmatrix} -1 & -2 \\ -3 & -4 \end{pmatrix}$ (D) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

12. If $A = \begin{pmatrix} 4 & -2 \\ 6 & -3 \end{pmatrix}$, then A^2 is

(A) $\begin{pmatrix} 16 & 4 \\ 36 & 9 \end{pmatrix}$ (B) $\begin{pmatrix} 8 & -4 \\ 12 & -6 \end{pmatrix}$ (C) $\begin{pmatrix} -4 & 2 \\ -6 & 3 \end{pmatrix}$ (D) $\begin{pmatrix} 4 & -2 \\ 6 & -3 \end{pmatrix}$

13. A is of order $m \times n$ and B is of order $p \times q$, addition of A and B is possible only if

(A) $m = p$ (B) $n = q$ (C) $n = p$ (D) $m = p, n = q$

14. If $\begin{pmatrix} a & 3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$, then the value of a is

(A) 8 (B) 4 (C) 2 (D) 11

15. If $A = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}$ is such that $A^2 = I$, then

(A) $1 + \alpha^2 + \beta\gamma = 0$ (B) $1 - \alpha^2 + \beta\gamma = 0$

(C) $1 - \alpha^2 - \beta\gamma = 0$ (D) $1 + \alpha^2 - \beta\gamma = 0$

16. If $A = [a_{ij}]_{2\times 2}$ and $a_{ij} = i + j$, then $A =$

(A) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ (B) $\begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$ (C) $\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$ (D) $\begin{pmatrix} 4 & 5 \\ 6 & 7 \end{pmatrix}$

17. $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, then the values of a, b, c and d respectively are

(A) $-1, 0, 0, -1$ (B) 1, 0, 0, 1 (C) $-1, 0, 1, 0$ (D) 1, 0, 0, 0

18. If $A = \begin{pmatrix} 7 & 2 \\ 1 & 3 \end{pmatrix}$ and $A + B = \begin{pmatrix} -1 & 0 \\ 2 & -4 \end{pmatrix}$, then the matrix $B =$

(A) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (B) $\begin{pmatrix} 6 & 2 \\ 3 & -1 \end{pmatrix}$ (C) $\begin{pmatrix} -8 & -2 \\ 1 & -7 \end{pmatrix}$ (D) $\begin{pmatrix} 8 & 2 \\ -1 & 7 \end{pmatrix}$

19. If $\begin{pmatrix} 5 & x & 1 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 20 \end{pmatrix}$, then the value of x is

(A) 7 (B) -7 (C) $\frac{1}{7}$ (D) 0

20. Which one of the following is true for any two square matrices A and B of same order?.

(A) $(AB)^T = A^T B^T$ (B) $(A^T B)^T = A^T B^T$ (C) $(AB)^T = BA$ (D) $(AB)^T = B^T A^T$

Points to Remember

- A matrix is a rectangular array of numbers.
- A matrix having m rows and n columns, is of the order $m \times n$.
- $A = [a_{ij}]_{m \times n}$ is a row matrix if $m = 1$.
- $A = [a_{ij}]_{m \times n}$ is a column matrix if $n = 1$.
- $A = [a_{ij}]_{m \times n}$ is a square matrix if $m = n$.
- $A = [a_{ij}]_{n \times n}$ is diagonal matrix if $a_{ij} = 0$, when $i \neq j$.
- $A = [a_{ij}]_{n \times n}$ is a scalar matrix if $a_{ij} = 0$, when $i \neq j$ and $a_{ij} = k$, when $i = j$. (k is a non-zero constant).
- $A = [a_{ij}]$ is unit matrix if $a_{ij} = 1$, when $i = j$ and $a_{ij} = 0$, when $i \neq j$.
- A matrix is said to be a zero matrix if all its elements are zero.
- Two matrices A and B are equal if the matrices A and B are of same order and their corresponding entries are equal.
- Addition or subtraction of two matrices are possible only when they are of same order.
- Matrix addition is commutative.
  That is, $A + B = B + A$, if A and B are matrices of same order.
- Matrix addition is Associative.
  That is, $(A + B) + C = A + (B + C)$, if A, B and C are matrices of same order.
- If A is a matrix of order $m \times n$ and B is a matrix of order $n \times p$, then the product matrix AB is defined and is of order $m \times p$.
- Matrix multiplication is not commutative in general. i.e., $AB \neq BA$.
- Matrix multiplication is associative. i.e., $(AB)C = A(BC)$, if both sides are defined.
- $(A^T)^T = A$, $(A + B)^T = A^T + B^T$ and $(AB)^T = B^T A^T$.
- Matrices A and B are multiplicative inverses to each other if $AB = BA = I$.
- If $AB = O$, it is not necessary that $A = O$ or $B = O$.
  That is, product of two non-zero matrices may be a zero matrix.

Do you know?

The Abel Prize , which was awarded for the first time in 2003, amounts to One Million US dollar. It is an International Prize awarded by Norwegian Academy of Science and presented annually by the King of Norway to one or more outstanding Mathematicians.

S.R. Srinivasa Varadhan, an Indian-American Mathematician born in Chennai, was awarded the Abel Prize in 2007 for his fundamental contributions to Probability Theory and in particular for creating a unified theory of large deviations.

5

COORDINATE GEOMETRY

No human investigation can be called real science if it cannot be demonstrated mathematically - Leonardo de Vinci

- Introduction
- Section Formula
- Area of Triangle and Quadrilateral
- Straight Lines

Pierre de Fermat
(1601-1665)
France

Together with Rene Descartes, Fermat was one of the two leading mathematicians of the first half of the 17th century. He discovered the fundamental principles of analytical geometry. He discovered an original method of finding the greatest and the smallest ordinates of curved lines.

He made notable contributions to coordinate geometry. Fermat's pioneering work in analytic geometry was circulated in manuscript form in 1636, predating the publication of Descarte's famous "La geometrie".

5.1 Introduction

Coordinate geometry, also known as analytical geometry is the study of geometry using a coordinate system and the principles of algebra and analysis. It helps us to interpret algebraic results geometrically and serves as a bridge between algebra and geometry. A systematic study of geometry using algebra was carried out by a French philosopher and a mathematician Rene Descartes. The use of coordinates was Descartes's great contribution to mathematics, which revolutionized the study of geometry. He published his book "La Geometry" in 1637. In this book, he converted a geometric problem into an algebraic equation, simplified and then solved the equation geometrically. French mathematician Pierre De Fermat also formulated the coordinate geometry at the same period and made great contribution to this field. In 1692, a German mathematician Gottfried Wilhelm Von Leibnitz introduced the modern terms like abscissa and ordinate in coordinate geometry . According to Nicholas Murray Butler, "The analytical geometry of Descartes and the calculus of Newton and Leibntiz have expanded into the marvelous mathematical method".

In class IX, we have studied the basic concepts of the coordinate geometry namely, the coordinate axes, plane, plotting of points in a plane and the distance between two points. In this chapter, we shall study about section formula, area of a triangle, slope and equation of a straight line.

5.2 Section formula

Let us look at the following problem.

Let A and B be two towns. Assume that one can reach town B from A by moving 60km towards east and then 30km towards north . A telephone company wants to raise a relay tower at

P which divides the line joining A and B in the ratio $1:2$ internally. Now, it wants to find the position of P where the relay tower is to be set up.

Choose the point A as the origin. Let $P(x,y)$ be the point. Draw the perpendiculars from P and B to the x-axis, meeting it in C and D respectively. Also draw a perpendicular from P to BD, intersecting at E.

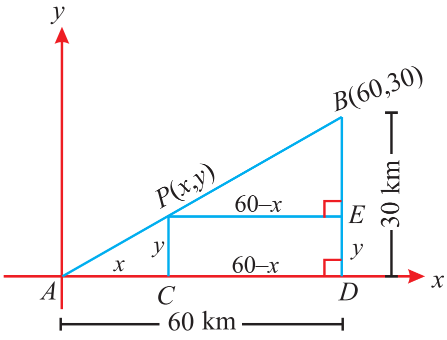


Fig. 5.1

Since ΔPAC and ΔBPE are similar, we have

$$\frac{AC}{PE} = \frac{PC}{BE} = \frac{AP}{PB} = \frac{1}{2}$$

Now $\quad \dfrac{AC}{PE} = \dfrac{1}{2}$

$$\Longrightarrow \frac{x}{60-x} = \frac{1}{2}$$

$$2x = 60 - x$$

Thus, $\quad x = 20$.

Also, $\quad \dfrac{PC}{BE} = \dfrac{1}{2}$

$$\Longrightarrow \frac{y}{30-y} = \frac{1}{2}$$

Thus, $\quad 2y = 30 - y \Longrightarrow y = 10$.

$\therefore$ The position of the relay tower is at $P(20,10)$.

Taking the above problem as a model, we shall derive the general section formula.

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two distinct points such that a point $P(x, y)$ divides AB internally in the ratio $l:m$. That is, $\dfrac{AP}{PB} = \dfrac{l}{m}$

From the Fig. 5.2, we get

$$AF = CD = OD - OC = x - x_1$$

$$PG = DE = OE - OD = x_2 - x$$

Also, $\quad PF = PD - FD = y - y_1$

$$BG = BE - GE = y_2 - y$$

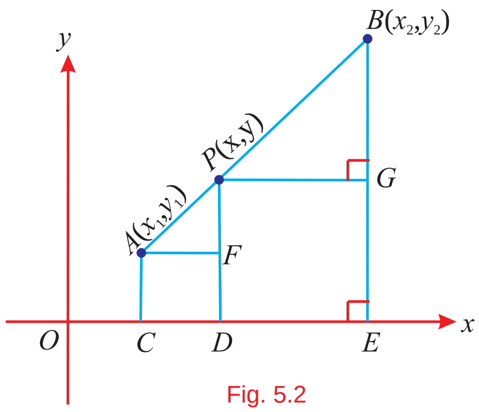


Fig. 5.2

Now, $\quad \Delta AFP$ and ΔPGB are similar.

(Refer chapter 6, section 6.3)

Thus, $\quad \dfrac{AF}{PG} = \dfrac{PF}{BG} = \dfrac{AP}{PB} = \dfrac{l}{m}$

$\therefore \quad \dfrac{AF}{PG} = \dfrac{l}{m}$ and $\dfrac{PF}{BG} = \dfrac{l}{m}$

$$\Longrightarrow \frac{x - x_1}{x_2 - x} = \frac{l}{m}$$

$$\Longrightarrow mx - mx_1 = lx_2 - lx$$

$$lx + mx = lx_2 + mx_1$$

$$\Longrightarrow x = \frac{lx_2 + mx_1}{l+m}$$

$$\Longrightarrow \frac{y - y_1}{y_2 - y} = \frac{l}{m}$$

$$\Longrightarrow my - my_1 = ly_2 - ly$$

$$ly + my = ly_2 + my_1$$

$$\Longrightarrow y = \frac{ly_2 + my_1}{l+m}$$

Thus, the point P which divides the line segment joining the two points $A(x_1, y_1)$ and $B(x_2, y_2)$ internally in the ratio $l:m$ is

$$P\left(\frac{lx_2 + mx_1}{l+m}, \frac{ly_2 + my_1}{l+m}\right)$$

This formula is known as **section formula.**

It is clear that the section formula can be used only when the related three points are collinear.

Results

(i) If P divides a line segment AB joining the two points $A(x_1, y_1)$ and $B(x_2, y_2)$ externally in the ratio $l:m$, then the point P is $\left(\frac{lx_2 - mx_1}{l-m}, \frac{ly_2 - my_1}{l-m}\right)$. In this case $\frac{l}{m}$ is negative.

(ii) **Midpoint of AB**

If M is the midpoint of AB, then M divides the line segment AB internally in the ratio 1:1. By substituting $l = 1$ and $m = 1$ in the section formula, we obtain

the midpoint of AB as $M\left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2}\right)$.

The midpoint of the line segment joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

(iii) **Centroid of a triangle**

Consider a ΔABC whose vertices are $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$. Let AD, BE and CF be the medians of the ΔABC.

We know that the medians of a triangle are concurrent and the point of concurrency is the centroid.

Let $G(x, y)$ be the centroid of ΔABC.

Now the midpoint of BC is $D\left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}\right)$

By the property of triangle, the centroid G divides the median AD internally in the ratio $2:1$

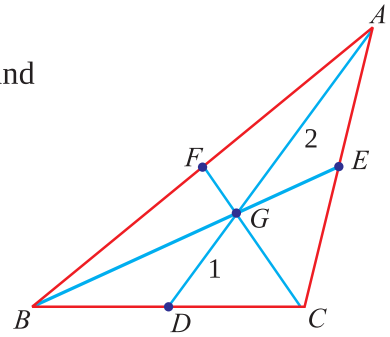


Fig. 5.3

$\therefore$ By section formula, the centroid

$$G(x, y) = G\left(\frac{2\frac{(x_2 + x_3)}{2} + 1(x_1)}{2+1}, \frac{2\frac{(y_2 + y_3)}{2} + 1(y_1)}{2+1}\right)$$

$$= G\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

The centroid of the triangle whose vertices are $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3), is $\left(\dfrac{x_1+x_2+x_3}{3}, \dfrac{y_1+y_2+y_3}{3}\right)$.

Example 5.1

Find the midpoint of the line segment joining the points $(3, 0)$ and $(-1, 4)$.

Solution Midpoint $M(x, y)$ of the line segment joining the points (x_1, y_1) and (x_2, y_2) is

$$M(x, y) = M\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

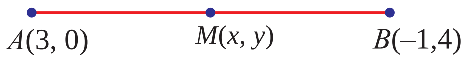


Fig. 5.4

$\therefore$ Midpoint of the line segment joining the points $(3, 0)$ and $(-1, 4)$ is

$$M(x, y) = \left(\frac{3-1}{2}, \frac{0+4}{2}\right) = M(1, 2).$$

Example 5.2

Find the point which divides the line segment joining the points (3 , 5) and (8 , 10) internally in the ratio 2 : 3.

Solution Let $A(3, 5)$ and $B(8, 10)$ be the given points.

Let the point $P(x,y)$ divide the line AB internally in the ratio 2 :3.

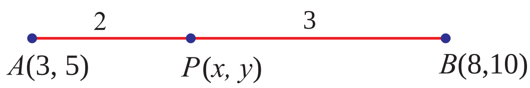


Fig. 5.5

By section formula, $P(x, y) = P\left(\dfrac{lx_2+mx_1}{l+m}, \dfrac{ly_2+my_1}{l+m}\right)$

Here $x_1 = 3, y_1 = 5, x_2 = 8, y_2 = 10$ and $l = 2, m = 3$

$$\therefore \quad P(x, y) = P\left(\frac{2(8)+3(3)}{2+3}, \frac{2(10)+3(5)}{2+3}\right) = P(5, 7)$$

Example 5.3

In what ratio does the point $P(-2, 3)$ divide the line segment joining the points $A(-3, 5)$ and $B(4, -9)$ internally?

Solution Given points are $A(-3, 5)$ and $B(4, -9)$.

Let $P(-2, 3)$ divide AB internally in the ratio $l : m$

By the section formula,

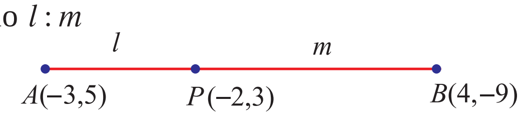


Fig. 5.6

$$P\left(\frac{lx_2+mx_1}{l+m}, \frac{ly_2+my_1}{l+m}\right) = P(-2, 3) \quad (1)$$

Here $x_1 = -3, y_1 = 5, x_2 = 4, y_2 = -9$.

(1) $\Longrightarrow \left(\dfrac{l(4)+m(-3)}{l+m}, \dfrac{l(-9)+m(5)}{l+m}\right) = (-2, 3)$

Equating the x-coordinates, we get

$$\frac{4l-3m}{l+m} = -2$$

$$\Longrightarrow \qquad 6l = m$$

$$\frac{l}{m} = \frac{1}{6}$$

i.e., $\qquad l : m = 1 : 6$

Hence P divides AB internally in the ratio 1 : 6

Note

(i) In the above example, one may get the ratio by equating y-coordinates also.

(ii) The ratios obtained by equating x-coordinates and by equating y-coordinates are same only when the three points are collinear.

(ii) If a point divides the line segment internally in the ratio $l:m$, then $\dfrac{l}{m}$ is positive.

(iii) If a point divides the line segment externally in the ratio $l:m$, then $\dfrac{l}{m}$ is negative.

Example 5.4

Find the points of trisection of the line segment joining $(4,-1)$ and $(-2,-3)$.

Solution Let $A(4,-1)$ and $B(-2,-3)$ be the given points.

Let $P(x,y)$ and $Q(a,b)$ be the points of trisection of AB so that $AP = PQ = QB$

Fig. 5.7

Hence P divides AB internally in the ratio 1 : 2 and Q divides AB internally in the ratio 2 : 1

Fig. 5.8

$\therefore$ By the section formula, the required points are

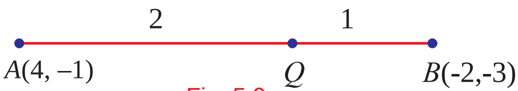


Fig. 5.9

$P\left(\dfrac{1(-2)+2(4)}{1+2}, \dfrac{1(-3)+2(-1)}{1+2}\right)$ and

$Q\left(\dfrac{2(-2)+1(4)}{2+1}, \dfrac{2(-3)+1(-1)}{2+1}\right)$

$\Longrightarrow \quad P(x,y) = P\left(\dfrac{-2+8}{3}, \dfrac{-3-2}{3}\right)$ and $\quad Q(a,b) = Q\left(\dfrac{-4+4}{3}, \dfrac{-6-1}{3}\right)$

$\qquad = P\left(2, -\dfrac{5}{3}\right) \qquad\qquad = Q\left(0, -\dfrac{7}{3}\right).$

Note that Q is the midpoint of PB and P is the midpoint of AQ.

Example 5.5

Find the centroid of the triangle whose vertices are $A(4, -6)$, $B(3, -2)$ and $C(5, 2)$.

Solution The centroid $G(x, y)$ of a triangle whose vertices are

(x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by

$$G(x, y) = G\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right).$$

We have $(x_1, y_1) = (4, -6)$, $(x_2, y_2) = (3, -2)$, $(x_3, y_3) = (5, 2)$

$\therefore$ The centroid of the triangle whose vertices are $(4, -6), (3, -2)$ and $(5, 2)$ is

$$G(x, y) = G\left(\frac{4 + 3 + 5}{3}, \frac{-6 - 2 + 2}{3}\right)$$

$$= G(4, -2).$$

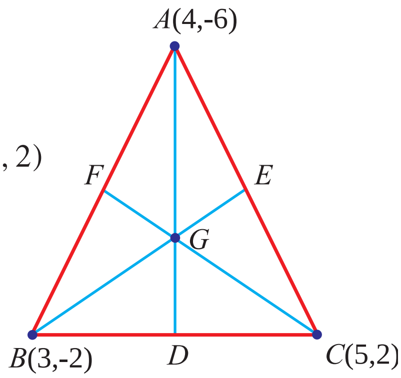


Fig. 5.10

Example 5.6

If $(7, 3), (6, 1), (8, 2)$ and $(p, 4)$ are the vertices of a parallelogram taken in order, then find the value of p.

Solution Let the vertices of the parallelogram be $A(7, 3), B(6, 1), C(8, 2)$ and $D(p, 4)$.

We know that the diagonals of a parallelogram bisect each other.

$\therefore$ The midpoints of the diagonal AC and the diagonal BD coincide.

Hence $$\left(\frac{7+8}{2}, \frac{3+2}{2}\right) = \left(\frac{6+p}{2}, \frac{1+4}{2}\right)$$

$$\Longrightarrow \left(\frac{6+p}{2}, \frac{5}{2}\right) = \left(\frac{15}{2}, \frac{5}{2}\right)$$

Equating the x-coordinates, we get,

$$\frac{6+p}{2} = \frac{15}{2}$$

$$\therefore p = 9$$

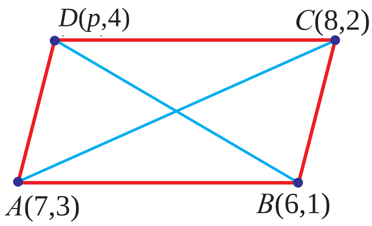


Fig. 5.11

Example 5.7

If C is the midpoint of the line segment joining $A(4, 0)$ and $B(0, 6)$ and if O is the origin, then show that C is equidistant from all the vertices of $\triangle OAB$.

Solution The midpoint of AB is $C\left(\frac{4+0}{2}, \frac{0+6}{2}\right) = C(2, 3)$

We know that the distance between $P(x_1, y_1)$ and $Q(x_2, y_2)$ is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

Distance between $O(0, 0)$ and $C(2, 3)$ is

$$OC = \sqrt{(2-0)^2 + (3-0)^2} = \sqrt{13}.$$

Distance between $A(4,0)$ and $C(2,3)$,

$$AC = \sqrt{(2-4)^2+(3-0)^2} = \sqrt{4+9} = \sqrt{13}$$

Distance between $B\ (0,6)$ and $C(2,3)$,

$$BC = \sqrt{(2-0)^2+(3-6)^2} = \sqrt{4+9} = \sqrt{13}$$

$\therefore \quad OC = AC = BC$

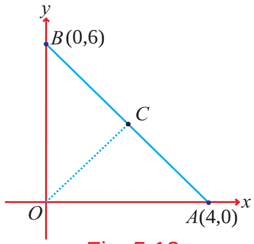


Fig. 5.12

$\therefore$ The point C is equidistant from all the vertices of the $\triangle OAB$.

Note The midpoint C of the hypotenuse, is the circumcentre of the right angled $\triangle OAB$.

Exercise 5.1

1. Find the midpoint of the line segment joining the points
   (i) $(1,-1)$ and $(-5,3)$ (ii) $(0,0)$ and $(0,4)$
2. Find the centroid of the triangle whose vertices are
   (i) $(1,3),(2,7)$ and $(12,-16)$ (ii) $(3,-5),(-7,4)$ and $(10,-2)$
3. The centre of a circle is at $(-6, 4)$. If one end of a diameter of the circle is at the origin, then find the other end.
4. If the centroid of a triangle is at (1, 3) and two of its vertices are $(-7, 6)$ and (8, 5) then find the third vertex of the triangle .
5. Using the section formula, show that the points $A(1,0)$, $B(5,3)$, $C(2,7)$ and $D(-2, 4)$ are the vertices of a parallelogram taken in order.
6. Find the coordinates of the point which divides the line segment joining (3, 4) and (–6, 2) in the ratio 3 : 2 externally.
7. Find the coordinates of the point which divides the line segment joining $(-3, 5)$ and $(4, -9)$ in the ratio 1 : 6 internally.
8. Let $A\ (-6,-5)$ and $B\ (-6, 4)$ be two points such that a point P on the line AB satisfies $AP = \frac{2}{9}AB$. Find the point P.
9. Find the points of trisection of the line segment joining the points $A(2,-2)$ and $B(-7, 4)$.
10. Find the points which divide the line segment joining $A(-4,0)$ and $B\ (0,6)$ into four equal parts.
11. Find the ratio in which the x-axis divides the line segment joining the points (6, 4) and $(1,-7)$.
12. In what ratio is the line joining the points $(-5, 1)$ and (2 , 3) divided by the y-axis? Also, find the point of intersection .
13. Find the length of the medians of the triangle whose vertices are $(1, -1)$, (0, 4) and $(-5, 3)$.

5.3 Area of a triangle

We have already learnt how to calculate the area of a triangle, when some measurements of the triangle are given. Now, if the coordinates of the vertices of a triangle are given, can we find its area ?

Let ABC be a triangle whose vertices are $A(x_1, y_1)$, $B(x_2, y_2)$, and $C(x_3, y_3)$.

Draw the lines AD, BE and CF perpendicular to x-axis.

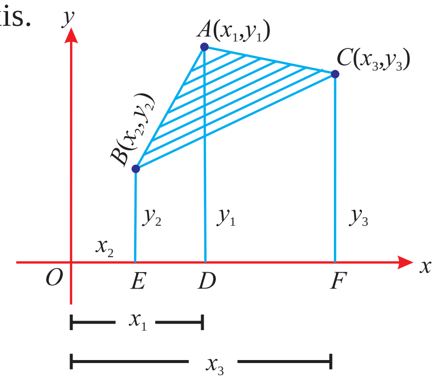


Fig. 5.13

From the figure, $ED = x_1 - x_2$, $DF = x_3 - x_1$ and

$EF = x_3 - x_2$.

Area of the triangle ABC

= Area of the trapezium $ABED$
+ Area of the trapezium $ADFC$
− Area of the trapezium $BEFC$

$$= \frac{1}{2}(BE + AD)ED + \frac{1}{2}(AD + CF)DF - \frac{1}{2}(BE + CF)EF$$

$$= \frac{1}{2}(y_2 + y_1)(x_1 - x_2) + \frac{1}{2}(y_1 + y_3)(x_3 - x_1) - \frac{1}{2}(y_2 + y_3)(x_3 - x_2)$$

$$= \frac{1}{2}\{x_1y_2 - x_2y_2 + x_1y_1 - x_2y_1 + x_3y_1 - x_1y_1 + x_3y_3 - x_1y_3 - x_3y_2 + x_2y_2 - x_3y_3 + x_2y_3\}$$

$\therefore$ Area of the ΔABC is $\frac{1}{2}\{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\}$.sq.units.

> If $A(x_1, y_1)$, $B(x_2, y_2)$, and $C(x_3, y_3)$ are the vertices of a ΔABC,
> then the area of the ΔABC is $\frac{1}{2}\{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\}$.sq.units.

Note

The area of the triangle can also be written as

$$\frac{1}{2}\{x_1y_2 - x_1y_3 + x_2y_3 - x_2y_1 + x_3y_1 - x_3y_2\} \text{ sq.units.}$$

(or) $$\frac{1}{2}\left\{(x_1y_2 + x_2\,y_3 + x_3y_1) - (x_2y_1 + x_3y_2 + x_1y_3)\right\}\text{sq.units}$$

The following pictorial representation helps us to write the above formula very easily.

Take the vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ of ΔABC in counter clockwise direction and write them column-wise as shown below.

$$\frac{1}{2}\begin{Bmatrix} x_1 & x_2 & x_3 & x_1 \\ y_1 & y_2 & y_3 & y_1 \end{Bmatrix}$$

Add the diagonal products x_1y_2, x_2y_3 and x_3y_1 as shown in the dark arrows.

Also add the products x_2y_1, x_3y_2 and x_1y_3 as shown in the dotted arrows and then subtract the latter from the former to get the expression $\frac{1}{2}\left\{(x_1y_2 + x_2y_3 + x_3y_1) - (x_2y_1 + x_3y_2 + x_1y_3)\right\}$

Note

To find the area of a triangle, the following steps may be useful.

(i) Plot the points in a rough diagram.

(ii) Take the vertices in counter clock-wise direction. Otherwise the formula gives a negative value.

(iii) Use the formula, area of the $\Delta ABC = \frac{1}{2}\left\{(x_1y_2 + x_2y_3 + x_3y_1) - (x_2y_1 + x_3y_2 + x_1y_3)\right\}$

5.4 Collinearity of three points

Three or more points in a plane are said to be collinear, if they lie on the same straight line.

In other words, three points $A(x_1,y_1)$, $B(x_2,y_2)$ and $C(x_3,y_3)$ are collinear if any one of these points lies on the straight line joining the other two points.

Suppose that the three points $A(x_1,y_1)$, $B(x_2,y_2)$ and $C(x_3,y_3)$ are collinear. Then they cannot form a triangle. Hence the area of the $\triangle ABC$ is zero.

i.e., $\frac{1}{2}\{(x_1y_2 + x_2y_3 + x_3y_1) - (x_2y_1 + x_3y_2 + x_1y_3)\} = 0$

$$\Longrightarrow \quad x_1y_2 + x_2y_3 + x_3y_1 = x_2y_1 + x_3y_2 + x_1y_3$$

One can prove that the converse is also true.

Hence the area of $\triangle ABC$ is zero if and only if the points *A*, *B* and *C* are collinear.

5.5 Area of the Quadrilateral

Let $A(x_1,y_1)$, $B(x_2,y_2)$, $C(x_3,y_3)$ and $D(x_4,y_4)$ be the vertices of a quadrilateral *ABCD*.

Now the area of the quadrilateral *ABCD* = area of the ΔABD+area of the ΔBCD

$$= \frac{1}{2}\{(x_1y_2 + x_2y_4 + x_4y_1) - (x_2y_1 + x_4y_2 + x_1y_4)\}$$
$$+ \frac{1}{2}\{(x_2y_3 + x_3y_4 + x_4y_2) - (x_3y_2 + x_4y_3 + x_2y_4)\}$$

$\therefore$ Area of the quadrilateral *ABCD*

$$= \frac{1}{2}\{(x_1y_2 + x_2y_3 + x_3y_4 + x_4y_1) - (x_2y_1 + x_3y_2 + x_4y_3 + x_1y_4)\}$$

or

$$\frac{1}{2}\{(x_1 - x_3)(y_2 - y_4) - (x_2 - x_4)(y_1 - y_3)\} \text{ sq.units}$$

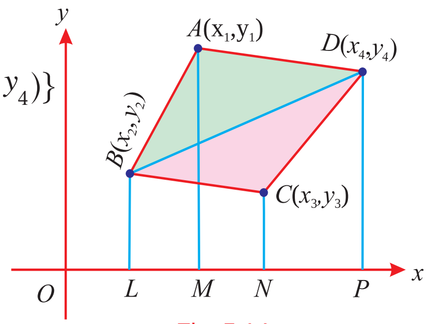


Fig. 5.14

The following pictorial representation helps us to write the above formula very easily.

Take the vertices $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ and $D(x_4, y_4)$ in counter clockwise direction and write them column-wise as shown below. Follow the same technique as we did in the case of finding the area of a triangle.

$$\frac{1}{2}\left\{\begin{matrix} x_1 & x_2 & x_3 & x_4 & x_1 \\ y_1 & y_2 & y_3 & y_4 & y_1 \end{matrix}\right\}.$$

This helps us to get the required expression.

Thus, the area of the quadrilateral ABCD

$$= \frac{1}{2}\{(x_1y_2 + x_2y_3 + x_3y_4 + x_4y_1) - (x_2y_1 + x_3y_2 + x_4y_3 + x_1y_4)\} \text{ sq. units.}$$

Example 5.8

Find the area of the triangle whose vertices are $(1, 2)$, $(-3, 4)$, and $(-5, -6)$.

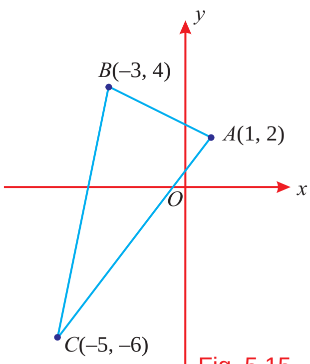


Fig. 5.15

Solution Plot the points in a rough diagram and take them in order.

Let the vertices be $A(1, 2)$, $B(-3, 4)$ and $C(-5, -6)$.

Now the area of $\triangle ABC$ is

$$= \frac{1}{2}\{(x_1y_2 + x_2y_3 + x_3y_1) - (x_2y_1 + x_3y_2 + x_1y_3)\}$$

$$= \frac{1}{2}\{(4 + 18 - 10) - (-6 - 20 - 6)\} \qquad \text{use : } \frac{1}{2}\left\{\begin{matrix} 1 & -3 & -5 & 1 \\ 2 & 4 & -6 & 2 \end{matrix}\right\}$$

$$= \frac{1}{2}\{12 + 32\} = 22. \text{ sq. units}$$

Example 5.9

If the area of the ΔABC is 68 sq.units and the vertices are $A(6, 7)$, $B(-4, 1)$ and $C(a, -9)$ taken in order, then find the value of a.

Solution Area of $\triangle ABC$ is

$$\frac{1}{2}\{(6 + 36 + 7a) - (-28 + a - 54)\} = 68 \qquad \text{use : } \frac{1}{2}\left\{\begin{matrix} 6 & -4 & a & 6 \\ 7 & 1 & -9 & 7 \end{matrix}\right\}$$

$$\Longrightarrow (42 + 7a) - (a - 82) = 136$$

$$\Longrightarrow 6a = 12 \qquad \therefore \quad a = 2$$

Example 5.10

Show that the points $A(2, 3)$, $B(4, 0)$ and $C(6, -3)$ are collinear.

Solution Area of the ΔABC is

$$= \frac{1}{2}\{(0 - 12 + 18) - (12 + 0 - 6)\} \qquad \text{use : } \frac{1}{2}\left\{\begin{matrix} 2 & 4 & 6 & 2 \\ 3 & 0 & -3 & 3 \end{matrix}\right\}$$

$$= \frac{1}{2}\{6 - 6\} = 0.$$

$\therefore$ The given points are collinear.

Example 5.11

If $P(x, y)$ is any point on the line segment joining the points $(a, 0)$ and $(0, b)$, then , prove that $\frac{x}{a} + \frac{y}{b} = 1$, where $a, b \neq 0$.

Solution Now the points (x, y), $(a, 0)$ and $(0, b)$ are collinear.

$\therefore$ The area of the triangle formed by them is zero.

$\Longrightarrow \quad ab - bx - ay = 0$ use: $\frac{1}{2}\left\{\begin{array}{cccc} a & 0 & x & a \\ 0 & b & y & 0 \end{array}\right\}$

$\therefore \quad bx + ay = ab$

Dividing by ab on both sides, we get,

$$\frac{x}{a} + \frac{y}{b} = 1, \quad \text{where} \quad a, b \neq 0$$

Example 5.12

Find the area of the quadrilateral formed by the points $(-4, -2)$, $(-3, -5)$, $(3, -2)$ and $(2, 3)$.

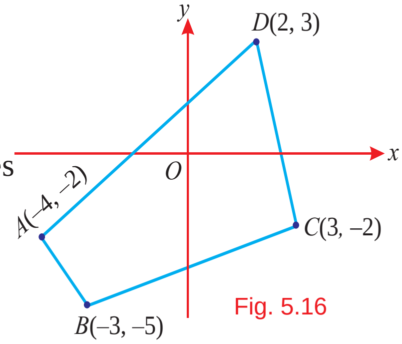


Fig. 5.16

Solution Let us plot the points roughly and take the vertices in counter clock-wise direction.

Let the vertices be

$A(-4, -2)$, $B(-3, -5)$, $C(3, -2)$ and $D(2, 3)$.

Area of the quadrilateral $ABCD$

$$= \frac{1}{2}\{(x_1y_2 + x_2y_3 + x_3y_4 + x_4y_1) - (x_2y_1 + x_3y_2 + x_4y_3 + x_1y_4)\}.$$

$$= \frac{1}{2}\{(20 + 6 + 9 - 4) - (6 - 15 - 4 - 12)\}$$

$$= \frac{1}{2}\{31 + 25\} = 28 \text{ sq.units.} \qquad \frac{1}{2}\left\{\begin{array}{ccccc} -4 & -3 & 3 & 2 & -4 \\ -2 & -5 & -2 & 3 & -2 \end{array}\right\}$$

Exercise 5.2

1. Find the area of the triangle formed by the points
   (i) (0, 0), (3, 0) and (0, 2)   (ii) (5, 2), (3, –5) and (–5, –1)
   (iii) (–4, –5), (4, 5) and (–1, –6)

2. Vertices of the triangles taken in order and their areas are given below. In each of the following find the value of a.

	Vertices	Area (in sq. units)
(i)	$(0, 0)$, $(4, a)$, $(6, 4)$	17
(ii)	(a, a), $(4, 5)$, $(6, -1)$	9
(iii)	$(a, -3)$, $(3, a)$, $(-1, 5)$	12

3. Determine if the following set of points are collinear or not.
   (i) (4, 3), (1, 2) and (−2, 1) (ii) (−2, −2), (−6, −2) and (−2, 2)
   (iii) $\left(-\frac{3}{2},3\right)$,(6, −2) and (−3, 4)

4. In each of the following, find the value of k for which the given points are collinear.
   (i) (k, −1), (2, 1) and (4, 5) (ii) $(2,-5),(3,-4)$ and $(9,k)$
   (iii) $(k,k),(2,3)$ and $(4,-1)$

5. Find the area of the quadrilateral whose vertices are
   (i) $(6,9),(7,4),(4,2)$ and $(3,7)$ (ii) $(-3,4),(-5,-6),(4,-1)$ and $(1,2)$
   (iii) $(-4,5),(0,7),(5,-5)$ and $(-4,-2)$

6. If the three points $(h,0),(a,b)$ and $(0,k)$ lie on a straight line, then using the area of the triangle formula, show that $\frac{a}{h}+\frac{b}{k}=1$, where $h,k\neq 0$.

7. Find the area of the triangle formed by joining the midpoints of the sides of a triangle whose vertices are $(0,-1),(2,1)$ and $(0,3)$. Find the ratio of this area to the area of the given triangle.

5.6 Straight Lines

5.6.1 Angle of Inclination

Let a straight line l intersect the x-axis at A. The angle between the positive x-axis and the line l, measured in counter clockwise direction is called the angle of inclination of the straight line l.

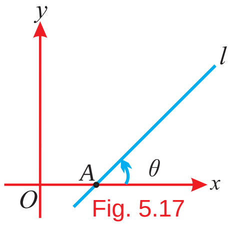


Fig. 5.17

Remarks

If θ is the angle of inclination of a straight line l, then

(i) $0^\circ \le \theta \le 180^\circ$

(ii) For horizontal lines, $\theta = 0^\circ$ or 180° and for vertical lines, $\theta = 90^\circ$

(iii) If a straight line initially lies along the x-axis and starts rotating about a fixed point A on the x-axis in the counter clockwise direction and finally coincides with the x-axis, then the angle of inclination of the straight line in the initial position is 0° and that of the line in the final position is 180°.

(iv) Lines which are perpendicular to x-axis are called as vertical lines. Other lines which are not perpendicular to x-axis are called as non vertical lines.

5.6.2 Slope of a straight line

Definition

If θ is the angle of inclination of a non-vertical straight line l, then $\tan\theta$ is called the Slope or Gradient of the line and is denoted by m.

$\therefore$ The slope of the straight line, $m = \tan\theta$ for $0^\circ \le \theta \le 180^\circ$, $\theta \neq 90^\circ$

Remarks

(i) Thus, the slope of x-axis or straight lines parallel to x-axis is zero.

(ii) The slope of y-axis or a straight line parallel to y-axis is not defined because $\tan 90^0$ is not defined. Therefore, whenever we talk about the slope of a straight line, we mean that of a non-vertical straight line.

(iii) If θ is acute, then the slope is positive, whereas if θ is obtuse then the slope is negative.

5.6.3 Slope of a straight line when any two points on the line are given

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be any two points on the straight line l whose angle of inclination is θ. Here, $0^\circ \leq \theta \leq 180^\circ$, $\theta \neq 90^\circ$

Let the straight line AB intersect the x-axis at C.

Now, the slope of the line l is $m = \tan\theta$ (1)

Draw AD and BE perpendicular to x-axis and draw the perpendicular AF line from A to BE.

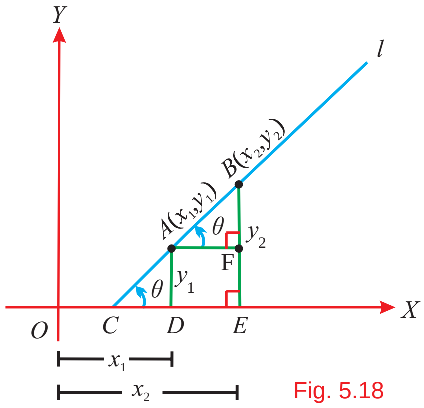


Fig. 5.18

From the figure, we have

$$AF = DE = OE - OD = x_2 - x_1$$

and $\quad BF = BE - EF = BE - AD = y_2 - y_1$

Also, we observe that $\angle DCA = \angle FAB = \theta$

In the right angled ΔABF, we have

$$\tan\theta = \frac{BF}{AF} = \frac{y_2 - y_1}{x_2 - x_1} \quad if\ x_1 \neq x_2 \qquad (2)$$

From (1) and (2), we get the slope, $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

The slope of the straight line joining the points $(\boldsymbol{x_1}, \boldsymbol{y_1})$ and $(\boldsymbol{x_2}, \boldsymbol{y_2})$ is

$$m = \frac{\boldsymbol{y_2 - y_1}}{\boldsymbol{x_2 - x_1}} = \frac{\boldsymbol{y_1 - y_2}}{\boldsymbol{x_1 - x_2}} \text{ where } x_1 \neq x_2 \text{ as } \theta \neq 90^\circ.$$

Note

The slope of the straight line joining the points (x_1, y_1) and (x_2, y_2) is also interpreted as

$$m = \frac{\boldsymbol{y_2 - y_1}}{\boldsymbol{x_2 - x_1}} = \frac{\text{change in } y \text{ coordinates}}{\text{change in } x \text{ coordinates}}.$$

5.6.4 Condition for parallel lines in terms of their slopes

Consider parallel lines l_1 and l_2 whose angles of inclination are θ_1 and θ_2 and slopes are m_1 and m_2 respectively.

Since l_1 and l_2 are parallel, the angles of inclinations θ_1 and θ_2 are equal.

$$\therefore \quad \tan\theta_1 = \tan\theta_2 \implies m_1 = m_2$$

$\therefore$ If two non-vertical straight lines are parallel, then their slopes are equal.

The converse is also true. i.e., if the slopes of two lines are equal, then the straight lines are parallel.

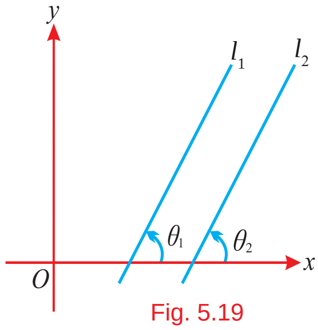


Fig. 5.19

5.6.5 Condition for perpendicular lines in terms of their slopes

Let l_1 and l_2 be two perpendicular straight lines passing through the points $A(x_1, y_1)$ and $B(x_2, y_2)$ respectively.

Let m_1 and m_2 be their slopes.

Let $C(x_3, y_3)$ be their point of intersection.

The slope of the straight line l_1 is $m_1 = \dfrac{y_3 - y_1}{x_3 - x_1}$

The slope of the straight line l_2 is $m_2 = \dfrac{y_3 - y_2}{x_3 - x_2}$

In the right angled $\triangle ABC$, we have

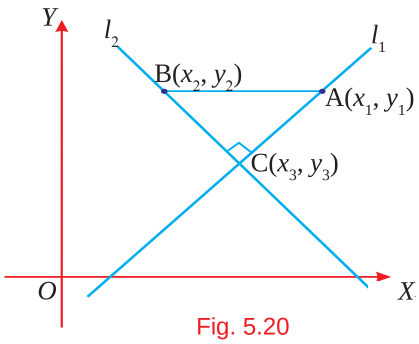


Fig. 5.20

$$AB^2 = AC^2 + BC^2$$

$$\implies (x_2 - x_1)^2 + (y_2 - y_1)^2 = (x_3 - x_1)^2 + (y_3 - y_1)^2 + (x_3 - x_2)^2 + (y_3 - y_2)^2$$

$$\implies (x_2 - x_3 + x_3 - x_1)^2 + (y_2 - y_3 + y_3 - y_1)^2$$
$$= (x_3 - x_1)^2 + (y_3 - y_1)^2 + (x_3 - x_2)^2 + (y_3 - y_2)^2$$

$$\implies \left(x_2 - x_3\right)^2 + (x_3 - x_1)^2 + 2(x_2 - x_3)(x_3 - x_1) + (y_2 - y_3)^2 + (y_3 - y_1)^2 + 2(y_2 - y_3)(y_3 - y_1)$$
$$= (x_3 - x_1)^2 + (y_3 - y_1)^2 + (x_3 - x_2)^2 + (y_3 - y_2)^2$$

$$\implies 2(x_2 - x_3)(x_3 - x_1) + 2(y_2 - y_3)(y_3 - y_1) = 0$$

$$\implies (y_2 - y_3)(y_3 - y_1) = -(x_2 - x_3)(x_3 - x_1)$$

$$\left(\frac{y_3 - y_1}{x_3 - x_1}\right)\left(\frac{y_3 - y_2}{x_3 - x_2}\right) = -1.$$

$$\implies m_1 m_2 = -1 \text{ or } m_1 = -\frac{1}{m_2}$$

If two non-vertical straight lines with slopes m_1 and m_2, are perpendicular, then

$$m_1 m_2 = -1.$$

On the other hand, if $m_1 m_2 = -1$, then the two straight lines are perpendicular.

Note

The straight lines x-axis and y-axis are perpendicular to each other. But, the condition $m_1 m_2 = -1$ is not true because the slope of the x-axis is zero and the slope of the y-axis is not defined.

Example 5.13

Find the angle of inclination of the straight line whose slope is $\frac{1}{\sqrt{3}}$.

Solution If θ is the angle of inclination of the line, then the slope of the line is

$$m = \tan\theta \quad \text{where } 0^\circ \le \theta \le 180^\circ, \ \theta \ne 90^\circ.$$

$$\therefore \ \tan\theta = \frac{1}{\sqrt{3}} \implies \theta = 30^\circ$$

Example 5.14

Find the slope of the straight line whose angle of inclination is 45°.

Solution If θ is the angle of inclination of the line, then the slope of the line is $m = \tan\theta$

Given that $m = \tan 45^\circ \implies m = 1$.

Example 5.15

Find the slope of the straight line passing through the points $(3, -2)$ and $(-1, 4)$.

Solution Slope of the straight line passing through the points (x_1, y_1) and (x_2, y_2) is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Slope of the straight line passing through the points (3 , –2) and (–1 , 4) is

$$m = \frac{4+2}{-1-3} = -\frac{3}{2}.$$

Example 5.16

Using the concept of slope, show that the points $A(5, -2)$, $B(4, -1)$ and $C(1, 2)$ are collinear.

Solution Slope of the line joining the points (x_1, y_1) and (x_2, y_2) is given by $m = \frac{y_2 - y_1}{x_2 - x_1}$

Slope of the line AB joining the points $A(5, -2)$ and $B(4 -1)$ is $m_1 = \frac{-1+2}{4-5} = -1$

Slope of the line BC joining the points $B(4,-1)$ and $C(1, 2)$ is $m_2 = \frac{2+1}{1-4} = -1$

Thus, slope of AB = slope of BC.

Also, B is the common point.

Hence, the points A, B and C are collinear.

Example 5.17

Using the concept of slope, show that the points $(-2, -1)$, $(4, 0)$, $(3, 3)$ and $(-3, 2)$ taken in order form a parallelogram.

Solution Let $A(-2, -1)$, $B(4, 0)$, $C(3, 3)$ and $D(-3, 2)$ be the given points taken in order.

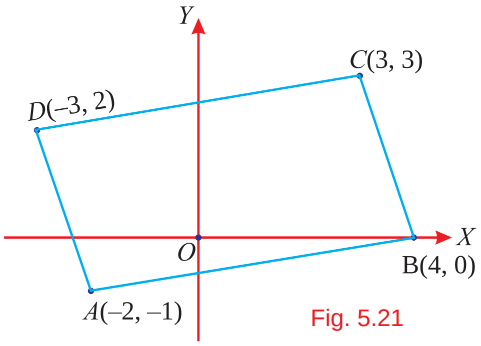


Fig. 5.21

Now the slope of $AB = \dfrac{0+1}{4+2} = \dfrac{1}{6}$

Slope of $CD = \dfrac{2-3}{-3-3} = \dfrac{1}{6}$

$\therefore$ Slope of AB = slope of CD

Hence, AB is parallel to CD. (1)

Now the slope of $BC = \dfrac{3-0}{3-4} = -3$

Slope of $AD = \dfrac{2+1}{-3+2} = -3$

$\therefore$ Slope of BC = slope of AD

Hence, BC is parallel to AD. (2)

From (1) and (2), we see that opposite sides of quadrilateral $ABCD$ are parallel

$\therefore$ $ABCD$ is a parallelogram.

Example 5.18

The vertices of a $\triangle ABC$ are $A(1, 2)$, $B(-4, 5)$ and $C(0, 1)$. Find the slopes of the altitudes of the triangle.

Solution Let AD, BE and CF be the altitudes of a $\triangle ABC$.

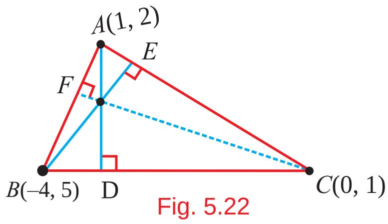


Fig. 5.22

slope of $BC = \dfrac{1-5}{0+4} = -1$

Since the altitude AD is perpendicular to BC,

slope of $AD = 1$ $\because m_1 m_2 = -1$

slope of $AC = \dfrac{1-2}{0-1} = 1$

Thus, slope of $BE = -1$ $\because BE \perp AC$

Also, slope of $AB = \dfrac{5-2}{-4-1} = -\dfrac{3}{5}$

$\therefore$ slope of $CF = \dfrac{5}{3}$ $\because CF \perp AB$

Exercise 5.3

1. Find the angle of inclination of the straight line whose slope is
   (i) 1 (ii) $\sqrt{3}$ (iii) 0
2. Find the slope of the straight line whose angle of inclination is
   (i) 30° (ii) 60° (iii) 90°
3. Find the slope of the straight line passing through the points
   (i) $(3\,,-2)$ and $(7\,,2)$ (ii) $(2\,,-4)$ and origin
   (iii) $(1+\sqrt{3}\,,2)$ and $(3+\sqrt{3}\,,4)$
4. Find the angle of inclination of the line passing through the points
   (i) $(1\,,2)$ and $(2\,,3)$ (ii) $(3\,,\sqrt{3})$ and $(0\,,0)$
   (iii) $(a\,,b)$ and $(-a\,,-b)$
5. Find the slope of the line which passes through the origin and the midpoint of the line segment joining the points $(0\,,-4)$ and $(8\,,0)$.
6. The side AB of a square $ABCD$ is parallel to x-axis . Find the
   (i) slope of AB (ii) slope of BC (iii) slope of the diagonal AC
7. The side BC of an equilateral $\triangle ABC$ is parallel to x-axis. Find the slope of AB and the slope of BC.
8. Using the concept of slope, show that each of the following set of points are collinear.
   (i) $(2\,,3)$, $(3\,,-1)$ and $(4\,,-5)$
   (ii) $(4\,,1)$, $(-2\,,-3)$ and $(-5\,,-5)$ (iii) $(4\,,4)$, $(-2\,,6)$ and $(1\,,5)$
9. If the points $(a, 1)$, $(1, 2)$ and $(0, b+1)$ are collinear, then show that $\frac{1}{a}+\frac{1}{b}=1$.
10. The line joining the points $A(-2\,,3)$ and $B(a\,,5)$ is parallel to the line joining the points $C(0\,,5)$ and $D(-2\,,1)$. Find the value of a.
11. The line joining the points $A(0, 5)$ and $B(4, 2)$ is perpendicular to the line joining the points $C(-1, -2)$ and $D(5, b)$. Find the value of b.
12. The vertices of $\triangle ABC$ are $A(1, 8)$, $B(-2, 4)$, $C(8, -5)$. If M and N are the midpoints of AB and AC respectively, find the slope of MN and hence verify that MN is parallel to BC.
13. A triangle has vertices at $(6\,,7)$, $(2\,,-9)$ and $(-4\,,1)$. Find the slopes of its medians.
14. The vertices of a $\triangle ABC$ are $A(-5\,,7)$, $B(-4\,,-5)$ and $C(4\,,5)$. Find the slopes of the altitudes of the triangle.

15. Using the concept of slope, show that the vertices $(1 , 2)$, $(-2 , 2)$, $(-4 , -3)$ and $(-1, -3)$ taken in order form a parallelogram.

16. Show that the opposite sides of a quadrilateral with vertices $A(-2 ,-4)$, $B(5 , -1)$, $C(6 , 4)$ and $D(-1, 1)$ taken in order are parallel.

5.6.6 Equation of a straight line

Let L be a straight line in the plane. A first degree equation $px + qy + r = 0$ in the variables x and y is satisfied by the x-coordinate and y-coordinate of any point on the line L and any values of x and y that satisfy this equation will be the coordinates of a point on the line L. Hence this equation is called the equation of the straight line L. We want to describe this line L algebraically. That is, we want to describe L by an algebraic equation. Now L is in any one of the following forms:

(i) horizontal line (ii) vertical line (iii) neither vertical nor horizontal

(i) Horizontal line: Let L be a horizontal line.

Then either L is x-axis or L is a horizontal line other than x-axis.

Case (a) If L is x – axis, then a point (x, y) lies on L if and only if $y = 0$ and x can be any real number.

Thus, $y = 0$ describes x – axis.

$\therefore$ The equation of x-axis is $y = 0$

Case (b) L is a horizontal line other than x-axis.

That is, L is parallel to x-axis.

Now, a point (x, y) lies on L if and only if the y-coordinate must remain a constant and x can be any real number.

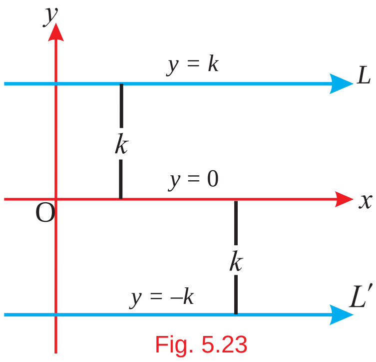


Fig. 5.23

$\therefore$ The equation of a straight line parallel to x-axis is $y = k$, where k is a constant.

Note that if $k > 0$, then L lies above x-axis and if $k < 0$, then L lies below x- axis. If $k = 0$, then L is nothing but the x-axis.

(ii) Vertical line: Let L be a vertical line.

Then either L is y-axis or L is a vertical line other than y-axis.

Case (a) If L is y-axis, then a point (x, y) in the plane lies on L if and only if $x = 0$ and y can be any real number.

Thus $x = 0$ describes y – axis.

$\therefore$ The equation of y-axis is $x = 0$

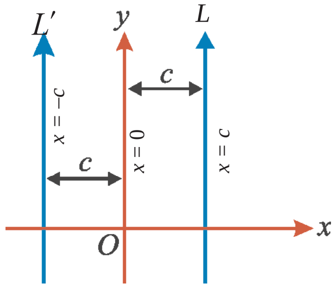


Fig. 5.24

Case (b) If L is a vertical line other than y-axis, then it is parallel to y-axis.

Now a point (x, y) lies on L if and only if x-coordinate must remain constant and y can be any real number.

$\therefore$ The equation of a straight line parallel to y-axis is $x = c$, where c is a constant.

Note that if $c > 0$, then L lies to the right y-axis and

if $c < 0$, then L lies to the left of y-axis.

If $c = 0$, then L is nothing but the y-axis.

(iii) Neither vertical nor horizontal: Let L be neither vertical nor horizontal.

In this case how do we describe L by an equation? Let θ denote the angle of inclination. Observe that if we know this θ and a point on L, then we can easily describe L.

> Slope m of a non-vertical line L can be calculated using
>
> (i) $m = \tan\theta$ if we know the angle of inclination θ.
>
> (ii) $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ if we know two distinct points (x_1, y_1), (x_2, y_2) on L.
>
> (iii) $m = 0$ if and only if L is horizontal.

Now consider the case where L is not a vertical line and derive the equation of a straight line in the following forms: (a) Slope-Point form (b) Two-Points form (c) Slope-Intercept form (d) Intercepts form

(a) Slope-Point form

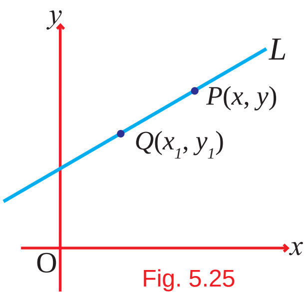


Fig. 5.25

Let m be the slope of L and $Q(x_1, y_1)$ be a point on L.

Let $P(x, y)$ be an arbitrary point on L other than Q. Then, we have

$$m = \frac{y - y_1}{x - x_1} \Leftrightarrow m(x - x_1) = y - y_1$$

Thus, the equation of a straight line with slope m and passing through (x_1, y_1) is

$y - y_1 = m(x - x_1)$ for all points (x, y) on L. (1)

Remarks

(i) Now the first degree equation (1) in the variables x and y is satisfied by the x-coordinate and y-coordinate of any point on the line L. Any value of x and y that satisfies this equation will be the coordinates of a point on the line L. Hence the equation (1) is called the equation of the straight line L.

(ii) The equation (1) says that the change in y-coordinates of the points on L is directly proportional to the change in x-coordinates. The proportionality constant m is the slope.

(b) Two-Points form

Suppose that two distinct points (x_1, y_1), (x_2, y_2) are given on a non-vertical line L.

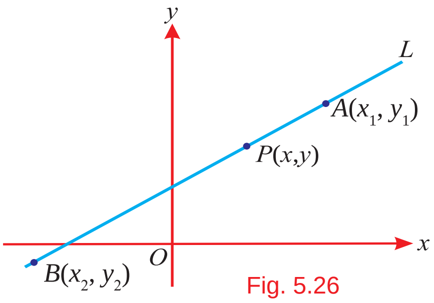


Fig. 5.26

To find the equation of L, we find the slope of L first and then use (1) .

The slope of L is

$$m = \frac{y_2 - y_1}{x_2 - x_1}, \text{ where } x_2 \neq x_1 \text{ as } L \text{ is non-vertical.}$$

Now, the formula (1) gives

$$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$$

$$\Longrightarrow \frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1} \quad \Longrightarrow \frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} \quad \text{for all points } (x, y) \text{ on } L \qquad (2)$$

Note

To get the equation of L, we can also use the point (x_2, y_2) instead of (x_1, y_1).

(c) Slope-Intercept form

Suppose that m is the slope of L and c is the y-intercept of L.

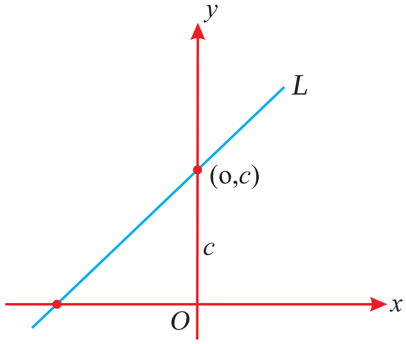


Fig. 5.27

Since c is the y-intercept, the point $(0, c)$ lies on L. Now using (1) with $(x_1, y_1) = (0, c)$ we obtain, $y - c = m(x - 0)$

$$\Longrightarrow \quad y = mx + c \quad \text{for all points } (x, y) \text{ on } L. \qquad (3)$$

Thus, $y = mx + c$ is the equation of straight line in the **Slope-Intercept form.**

(d) Intercepts form

Suppose that the straight line L makes non-zero intercepts a and b on the x-axis and on the y-axis respectively.

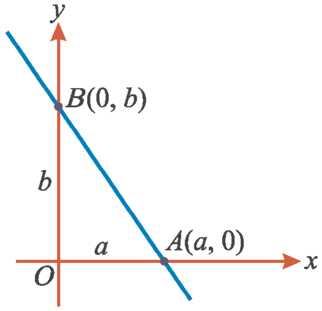


Fig. 5.28

$\therefore$ The straight line cuts the x-axis at $A(a, 0)$ and the y-axis at $B(0, b)$

The slope of AB is $m = -\frac{b}{a}$.

Now (1) gives, $\quad y - 0 = -\frac{b}{a}(x - a)$

$$\Longrightarrow \quad ay = -bx + ab$$

$$bx + ay = ab$$

Divide by ab to get $\frac{x}{a} + \frac{y}{b} = 1$

$\therefore$ Equation of a straight line having x-intercept a and y-intercept b is

$$\frac{x}{a} + \frac{y}{b} = 1 \quad \text{for all points } (x, y) \text{ on } L \qquad (4)$$

Note

(i) If the line L with slope m, makes x-intercept d, then the equation of the line is $y = m(x - d)$.

(ii) The straight line $y = mx$ passes through the origin.(both x and y-intercepts are zero for $m \neq 0$).

(iii) Equations (1), (2) and (4) can be simplified to slope-intercept form given by (3).

(iv) Each equation in (1), (2), (3) and (4) can be rewritten in the form $px + qy + r = 0$ for all points (x, y) on L, which is called the general form of equation of a straight line.

Example 5.19

Find the equations of the straight lines parallel to the coordinate axes and passing through the point $(3, -4)$.

Solution Let L and L' be the straight lines passing through the point $(3, -4)$ and parallel to x-axis and y-axis respectively.

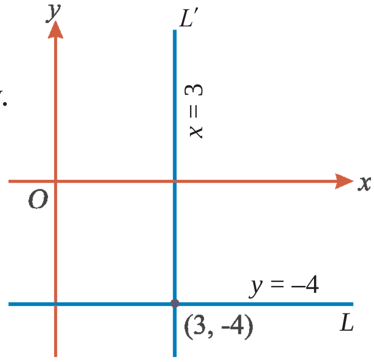


Fig. 5.29

The y-coordinate of every point on the line L is -4.

Hence, the equation of the line L is $\quad y = -4$

Similarly, the x-coordinate of every point on the straight line L' is 3

Hence, the equation of the line L' is $x = 3$.

Example 5.20

Find the equation of straight line whose angle of inclination is 45° and y-intercept is $\frac{2}{5}$.

Solution Slope of the line, $m = \tan\theta$

$$= \tan 45^\circ = 1$$

y-intercept is $c = \frac{2}{5}$

By the slope-intercept form, the equation of the straight line is

$$y = mx + c$$

$$y = x + \frac{2}{5} \implies y = \frac{5x+2}{5}$$

$\therefore$ The equation of the straight line is $5x - 5y + 2 = 0$

Example 5.21

Find the equation of the straight line passing through the point $(-2,3)$ with slope $\frac{1}{3}$.

Solution Given that the slope $m = \frac{1}{3}$ and a point $(x_1, y_1) = (-2, 3)$

By slope-point formula, the equation of the straight line is

$$y - y_1 = m(x - x_1)$$

$$\implies y - 3 = \frac{1}{3}(x+2)$$

Thus, $x - 3y + 11 = 0$ is the required equation.

Example 5.22

Find the equation of the straight line passing through the points $(-1,1)$ and $(2,-4)$.

Solution Let A(x_1, y_1) and $B(x_2, y_2)$ be the given points.

Here $x_1 = -1$, $y_1 = 1$ and $x_2 = 2$, $y_2 = -4$.

Using two-points formula, the equation of the straight line is

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

$$\implies \frac{y-1}{-4-1} = \frac{x+1}{2+1}$$

$$\implies 3y - 3 = -5x - 5$$

Hence, $5x + 3y + 2 = 0$ is the required equation of the straight line.

Example 5.23

The vertices of a $\triangle ABC$ are $A(2, 1)$, $B(-2, 3)$ and $C(4, 5)$. Find the equation of the median through the vertex A.

Solution Median is a straight line joining a vertex and the midpoint of the opposite side.

Let D be the midpoint of BC.

$\therefore$ Midpoint of BC is $D\left(\frac{-2+4}{2}, \frac{3+5}{2}\right) = D(1, 4)$

Now the equation of the median AD is

$$\frac{y-1}{4-1} = \frac{x-2}{1-2} \qquad \because (x_1, y_1) = (2, 1) \text{ and } (x_2, y_2) = (1, 4)$$

$$\frac{y-1}{3} = \frac{x-2}{-1}$$

$\therefore$ $3x + y - 7 = 0$ is the required equation.

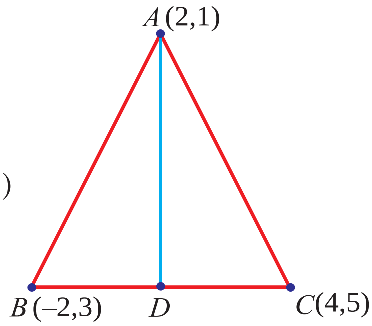


Fig. 5.30

Example 5.24

If the x-intercept and y-intercept of a straight line are $\frac{2}{3}$ and $\frac{3}{4}$ respectively, then find the equation of the straight line.

Solution Given that x-intercept of the straight line, $a = \frac{2}{3}$

and the y-intercept of the straight line, $b = \frac{3}{4}$

Using intercept form, the equation of the straight line is

$$\frac{x}{a} + \frac{y}{b} = 1 \implies \frac{x}{\frac{2}{3}} + \frac{y}{\frac{3}{4}} = 1$$

$$\implies \frac{3x}{2} + \frac{4y}{3} = 1$$

Hence, $9x + 8y - 6 = 0$ is the required equation.

Example 5.25

Find the equations of the straight lines each passing through the point $(6, -2)$ and whose sum of the intercepts is 5.

Solution Let a and b be the x-intercept and y-intercept of the required straight line respectively.

Given that sum of the intercepts, $a + b = 5$

$$\implies \qquad b = 5 - a$$

Now, the equation of the straight line in the intercept form is

$$\frac{x}{a} + \frac{y}{b} = 1 \implies \frac{x}{a} + \frac{y}{5-a} = 1$$

$$\implies \frac{(5-a)x + ay}{a(5-a)} = 1$$

Thus, $(5-a)x + ay = a(5-a)$ (1)

Since the straight line given by (1) passes through $(6, -2)$, we get,

$$(5-a)6+a(-2) = a(5-a)$$

$$\Longrightarrow \quad a^2-13a+30 = 0.$$

That is, $\quad (a-3)(a-10) = 0$

$\therefore \quad a = 3$ or $a = 10$

When $a = 3$, (1) $\Longrightarrow (5-3)x+3y = 3(5-3)$

$$\Longrightarrow \quad 2x+3y = 6 \qquad (2)$$

When $a = 10$, (1) $\Longrightarrow (5-10)x+10y = 10(5-10)$

$$\Longrightarrow \quad -5x+10y = -50$$

That is, $\quad x-2y-10 = 0. \qquad (3)$

Hence, $2x+3y = 6$ and $x-2y-10 = 0$ are the equations of required straight lines.

Exercise 5.4

1. Write the equations of the straight lines parallel to x- axis which are at a distance of 5 units from the x-axis.

2. Find the equations of the straight lines parallel to the coordinate axes and passing through the point $(-5,-2)$.

3. Find the equation of a straight line whose
   (i) slope is -3 and y-intercept is 4.
   (ii) angle of inclination is 60^0 and y-intercept is 3.

4. Find the equation of the line intersecting the y- axis at a distance of 3 units above the origin and $\tan\theta = \frac{1}{2}$, where θ is the angle of inclination.

5. Find the slope and y-intercept of the line whose equation is
   (i) $y = x+1$ (ii) $5x = 3y$ (iii) $4x-2y+1 = 0$ (iv) $10x+15y+6 = 0$

6. Find the equation of the straight line whose
   (i) slope is -4 and passing through $(1, 2)$
   (ii) slope is $\frac{2}{3}$ and passing through $(5, -4)$

7. Find the equation of the straight line which passes through the midpoint of the line segment joining $(4, 2)$ and $(3, 1)$ whose angle of inclination is 30^0.

8. Find the equation of the straight line passing through the points
   (i) $(-2, 5)$ and $(3, 6)$ (ii) $(0, -6)$ and $(-8, 2)$

9. Find the equation of the median from the vertex R in a $\triangle PQR$ with vertices at $P(1, -3)$, $Q(-2, 5)$ and $R(-3, 4)$.

10. By using the concept of the equation of the straight line, prove that the given three points are collinear.

    (i) (4, 2), (7, 5) and (9, 7) (ii) (1, 4), (3, −2) and (−3, 16)

11. Find the equation of the straight line whose x and y-intercepts on the axes are given by

    (i) 2 and 3 (ii) $-\frac{1}{3}$ and $\frac{3}{2}$ (iii) $\frac{2}{5}$ and $-\frac{3}{4}$

12. Find the x and y intercepts of the straight line

    (i) $5x + 3y - 15 = 0$ (ii) $2x - y + 16 = 0$ (iii) $3x + 10y + 4 = 0$

13. Find the equation of the straight line passing through the point (3, 4) and has intercepts which are in the ratio 3 : 2.

14. Find the equation of the straight lines passing through the point (2, 2) and the sum of the intercepts is 9.

15. Find the equation of the straight line passing through the point (5, −3) and whose intercepts on the axes are equal in magnitude but opposite in sign.

16. Find the equation of the line passing through the point (9, −1) and having its x-intercept thrice as its y-intercept.

17. A straight line cuts the coordinate axes at A and B. If the midpoint of AB is (3, 2), then find the equation of AB.

18. Find the equation of the line passing through (22, −6) and having intercept on x-axis exceeds the intercept on y-axis by 5.

19. If $A(3, 6)$ and $C(-1, 2)$ are two vertices of a rhombus $ABCD$, then find the equation of straight line that lies along the diagonal BD.

20. Find the equation of the line whose gradient is $\frac{3}{2}$ and which passes through P, where P divides the line segment joining $A(-2, 6)$ and $B(3, -4)$ in the ratio 2 : 3 internally.

5.7 General Form of Equation of a straight line

We have already pointed out that different forms of the equation of a straight line may be converted into the standard form $ax + by + c = 0$, where a, b and c are real constants such that either $a \neq 0$ or $b \neq 0$.

Now let us find out

(i) the slope of $ax + by + c = 0$

(ii) the equation of a straight line parallel to $ax + by + c = 0$

(iii) the equation of a straight line perpendicular to $ax + by + c = 0$ and

(iv) the point of intersection of two intersecting straight lines.

(i) The general form of the equation of a straight line is $ax+by+c=0$.

The above equation is rewritten as $y = -\frac{a}{b}x - \frac{c}{b},\ b \neq 0$ (1)

Comparing (1) with the slope-intercept form $y = mx + k$, we get,

$$\text{slope, } m = -\frac{a}{b} \text{ and the } y\text{-intercept } = -\frac{c}{b}$$

$\therefore$ For the equation $ax + by + c = 0$, we have

$$\text{slope } m = -\frac{\text{coefficient of } x}{\text{coefficient of } y} \text{ and the } y\text{-intercept is } -\frac{\text{constant term}}{\text{coefficient of } y}.$$

(ii) Equation of a line parallel to the line $ax+by+c=0$.

We know that two straight lines are parallel if and only if their slopes are equal.

Hence the equations of all lines parallel to the line $ax + by + c = 0$ are of the form

$$ax + by + k = 0, \text{ for different values of } k.$$

(iii) Equation of a line perpendicular to the line $ax+by+c=0$

We know that two non-vertical lines are perpendicular if and only if the product of their slopes is –1.

Hence the equations of all lines perpendicular to the line $ax + by + c = 0$ are

$$bx - ay + k = 0, \text{ for different values of } k.$$

Note

Two straight lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, where the coefficients are non-zero,

(i) are parallel if and only if $\frac{a_1}{a_2} = \frac{b_1}{b_2}$

(ii) are perpendicular if and only if $a_1a_2 + b_1b_2 = 0$

(iv) The point of intersection of two straight lines

If two straight lines are not parallel, then they will intersect at a point. This point lies on both the straight lines. Hence, the point of intersection is obtained by solving the given two equations.

Example 5.26

Show that the straight lines $3x + 2y - 12 = 0$ and $6x + 4y + 8 = 0$ are parallel.

Solution Slope of the straight line $3x + 2y - 12 = 0$ is $m_1 = -\frac{\text{coefficient of } x}{\text{coefficient of } y} = -\frac{3}{2}$

Similarly, the slope of the line $6x + 4y + 8 = 0$ is $m_2 = -\frac{6}{4} = -\frac{3}{2}$

$\therefore$ $m_1 = m_2$. Hence, the two straight lines are parallel.

Example 5.27

Prove that the straight lines $x+2y+1=0$ and $2x-y+5=0$ are perpendicular to each other.

Solution Slope of the straight line $x+2y+1=0$ is $m_1 = -\dfrac{\text{coefficient of } x}{\text{coefficient of } y} = -\dfrac{1}{2}$

Slope of the straight line $2x-y+5=0$ is $m_2 = -\dfrac{\text{coefficient of } x}{\text{coefficient of } y} = \dfrac{-2}{-1} = 2$

Product of the slopes $m_1 m_2 = -\dfrac{1}{2}\times 2 = -1$

$\therefore$ The two straight lines are perpendicular.

Example 5.28

Find the equation of the straight line parallel to the line $x-8y+13=0$ and passing through the point (2, 5).

Solution Equation of the straight line parallel to $x-8y+13=0$ is $x-8y+k=0$

Since it passes through the point (2, 5)

$$2-8(5)+k = 0 \implies k = 38$$

$\therefore$ Equation of the required straight line is $x-8y+38=0$

Example 5.29

The vertices of $\triangle ABC$ are $A(2, 1)$, $B(6, -1)$ and $C(4, 11)$. Find the equation of the straight line along the altitude from the vertex A.

Solution Slope of $BC = \dfrac{11+1}{4-6} = -6$

Since the line AD is perpendicular to the line BC, slope of $AD = \dfrac{1}{6}$

$\therefore$ Equation of AD is $y-y_1 = m(x-x_1)$

$$y-1 = \frac{1}{6}(x-2) \implies 6y-6 = x-2$$

$\therefore$ Equation of the required straight line is $x-6y+4=0$

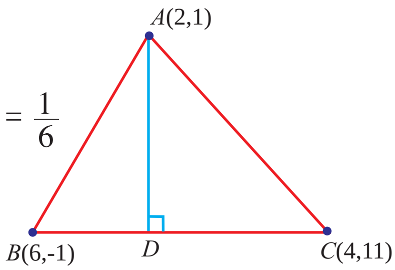


Fig. 5.31

Exercise 5.5

1. Find the slope of the straight line

   (i) $3x+4y-6=0$ (ii) $y=7x+6$ (iii) $4x=5y+3$.

2. Show that the straight lines $x+2y+1=0$ and $3x+6y+2=0$ are parallel.

3. Show that the straight lines $3x-5y+7=0$ and $15x+9y+4=0$ are perpendicular.

4. If the straight lines $\dfrac{y}{2} = x-p$ and $ax+5=3y$ are parallel, then find a.

5. Find the value of a if the straight lines $5x-2y-9=0$ and $ay+2x-11=0$ are perpendicular to each other.

6. Find the values of p for which the straight lines $8px+(2-3p)y+1=0$ and $px+8y-7=0$ are perpendicular to each other.

7. If the straight line passing through the points $(h,3)$ and (4, 1) intersects the line $7x-9y-19=0$ at right angle, then find the value of h.

8. Find the equation of the straight line parallel to the line $3x-y+7=0$ and passing through the point $(1,-2)$.

9. Find the equation of the straight line perpendicular to the straight line $x-2y+3=0$ and passing through the point $(1,-2)$.

10. Find the equation of the perpendicular bisector of the straight line segment joining the points (3, 4) and $(-1, 2)$.

11. Find the equation of the straight line passing through the point of intersection of the lines $2x+y-3=0$ and $5x+y-6=0$ and parallel to the line joining the points (1, 2) and (2, 1).

12. Find the equation of the straight line which passes through the point of intersection of the straight lines $5x-6y=1$ and $3x+2y+5=0$ and is perpendicular to the straight line $3x-5y+11=0$.

13. Find the equation of the straight line joining the point of intersection of the lines $3x-y+9=0$ and $x+2y=4$ and the point of intersection of the lines $2x+y-4=0$ and $x-2y+3=0$.

14. If the vertices of a $\triangle ABC$ are $A(2,-4)$, $B(3,3)$ and $C(-1,5)$. Find the equation of the straight line along the altitude from the vertex B.

15. If the vertices of a $\triangle ABC$ are $A(-4,4)$, $B(8,4)$ and $C(8,10)$. Find the equation of the straight line along the median from the vertex A.

16. Find the coordinates of the foot of the perpendicular from the origin on the straight line $3x+2y=13$.

17. If $x+2y=7$ and $2x+y=8$ are the equations of the lines of two diameters of a circle, find the radius of the circle if the point $(0,-2)$ lies on the circle.

18. Find the equation of the straight line segment whose end points are the point of intersection of the straight lines $2x-3y+4=0$, $x-2y+3=0$ and the midpoint of the line joining the points $(3,-2)$ and $(-5,8)$.

19. In an isosceles $\triangle PQR$, $PQ = PR$. The base QR lies on the x-axis, P lies on the y- axis and $2x-3y+9=0$ is the equation of PQ. Find the equation of the straight line along PR.

Exercise 5.6

Choose the correct answer

1. The midpoint of the line joining $(a, -b)$ and $(3a, 5b)$ is
   (A) $(-a, 2b)$ (B) $(2a, 4b)$ (C) $(2a, 2b)$ (D) $(-a, -3b)$
2. The point P which divides the line segment joining the points $A(1, -3)$ and $B(-3, 9)$ internally in the ratio 1:3 is
   (A) $(2, 1)$ (B) $(0, 0)$ (C) $\left(\frac{5}{3}, 2\right)$ (D) $(1, -2)$
3. If the line segment joining the points $A(3, 4)$ and $B(14, -3)$ meets the x-axis at P, then the ratio in which P divides the segment AB is
   (A) 4 : 3 (B) 3 : 4 (C) 2 : 3 (D) 4 : 1
4. The centroid of the triangle with vertices at $(-2, -5)$, $(-2, 12)$ and $(10, -1)$ is
   (A) $(6, 6)$ (B) $(4, 4)$ (C) $(3, 3)$ (D) $(2, 2)$
5. If $(1, 2)$, $(4, 6)$, $(x, 6)$ and $(3, 2)$ are the vertices of a parallelogram taken in order, then the value of x is
   (A) 6 (B) 2 (C) 1 (D) 3
6. Area of the triangle formed by the points (0,0), $(2, 0)$ and $(0, 2)$ is
   (A) 1 sq. units (B) 2 sq. units (C) 4 sq. units (D) 8 sq. units
7. Area of the quadrilateral formed by the points $(1, 1)$, $(0, 1)$, $(0, 0)$ and $(1, 0)$ is
   (A) 3 sq. units (B) 2 sq. units (C) 4 sq. units (D) 1 sq. units
8. The angle of inclination of a straight line parallel to x-axis is equal to
   (A) $0°$ (B) $60°$ (C) $45°$ (D) $90°$
9. Slope of the line joining the points $(3, -2)$ and $(-1, a)$ is $-\frac{3}{2}$, then the value of a is equal to
   (A) 1 (B) 2 (C) 3 (D) 4
10. Slope of the straight line which is perpendicular to the straight line joining the points $(-2, 6)$ and $(4, 8)$ is equal to
    (A) $\frac{1}{3}$ (B) 3 (C) -3 (D) $-\frac{1}{3}$
11. The point of intersection of the straight lines $9x - y - 2 = 0$ and $2x + y - 9 = 0$ is
    (A) $(-1, 7)$ (B) $(7, 1)$ (C) $(1, 7)$ (D) $(-1, -7)$
12. The straight line $4x + 3y - 12 = 0$ intersects the y- axis at
    (A) $(3, 0)$ (B) $(0, 4)$ (C) $(3, 4)$ (D) $(0, -4)$
13. The slope of the straight line $7y - 2x = 11$ is equal to
    (A) $-\frac{7}{2}$ (B) $\frac{7}{2}$ (C) $\frac{2}{7}$ (D) $-\frac{2}{7}$
14. The equation of a straight line passing through the point $(2, -7)$ and parallel to x-axis is
    (A) $x = 2$ (B) $x = -7$ (C) $y = -7$ (D) $y = 2$

15. The x and y-intercepts of the line $2x - 3y + 6 = 0$, respectively are

(A) 2, 3 (B) 3, 2 (C) $-3, 2$ (D) $3, -2$

16. The centre of a circle is $(-6, 4)$. If one end of the diameter of the circle is at $(-12, 8)$, then the other end is at

(A) $(-18, 12)$ (B) $(-9, 6)$ (C) $(-3, 2)$ (D) $(0, 0)$

17. The equation of the straight line passing through the origin and perpendicular to the straight line $2x + 3y - 7 = 0$ is

(A) $2x + 3y = 0$ (B) $3x - 2y = 0$ (C) $y + 5 = 0$ (D) $y - 5 = 0$

18. The equation of a straight line parallel to y-axis and passing through the point $(-2, 5)$ is

(A) $x - 2 = 0$ (B) $x + 2 = 0$ (C) $y + 5 = 0$ (D) $y - 5 = 0$

19. If the points (2, 5), (4, 6) and (a, a) are collinear, then the value of a is equal to

(A) -8 (B) 4 (C) -4 (D) 8

20. If a straight line $y = 2x + k$ passes through the point (1, 2), then the value of k is equal to

(A) 0 (B) 4 (C) 5 (D) -3

21. The equation of a straight line having slope 3 and y-intercept -4 is

(A) $3x - y - 4 = 0$ (B) $3x + y - 4 = 0$

(C) $3x - y + 4 = 0$ (D) $3x + y + 4 = 0$

22. The point of intersection of the straight lines $y = 0$ and $x = -4$ is

(A) $(0, -4)$ (B) $(-4, 0)$ (C) $(0, 4)$ (D) $(4, 0)$

23. The value of k if the straight lines $3x + 6y + 7 = 0$ and $2x + ky = 5$ are perpendicular is

(A) 1 (B) -1 (C) 2 (D) $\frac{1}{2}$

Points to Remember

- The distance between $P(x_1, y_1)$ and $Q(x_2, y_2)$ is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
- The point P which divides the line segment joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ internally in the ratio $l : m$ is $\left(\frac{lx_2 + mx_1}{l + m}, \frac{ly_2 + my_1}{l + m}\right)$.
- The point Q which divides the line segment joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ extrenally in the ratio $l : m$ is $\left(\frac{lx_2 - mx_1}{l - m}, \frac{ly_2 - my_1}{l - m}\right)$.
- Midpoint of the line segment joining the points (x_1, y_1) and (x_2, y_2) is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

- The area of the triangle formed by the points (x_1, y_1), (x_2, y_2) and (x_3, y_3) is
$$\frac{1}{2}\sum x_1(y_2 - y_3) = \frac{1}{2}\{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\}$$
$$= \frac{1}{2}\{(x_1y_2 + x_2y_3 + x_3y_1) - (x_2y_1 + x_3y_2 + x_1y_3)\}.$$
- Three points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are collinear if and only if
  (i) $x_1y_2 + x_2y_3 + x_3y_1 = x_2y_1 + x_3y_2 + x_1y_3$ (or)
  (ii) Slope of AB = Slope of BC or slope of AC.
- If a line makes an angle θ with the positive direction of x- axis, then the slope $m = \tan\theta$.
- Slope of the non-vertical line passing through the points (x_1, y_1) and (x_2, y_2) is
$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$$
- Slope of the line $ax + by + c = 0$ is $m = -\frac{\text{coefficient of } x}{\text{coefficient of } y} = -\frac{a}{b},\ b \neq 0$
- Slope of the horizontal line is 0 and slope of the vertical line is undefined.
- Two lines are parallel if and only if their slopes are equal.
- Two non-vertical lines are perpendicular if and only if the product of their slopes is -1. That is, $m_1m_2 = -1$.

Equation of straight lines

Sl.No	Straight line	Equation
1.	x-axis	$y = 0$
2.	y-axis	$x = 0$
3.	Parallel to x-axis	$y = k$
4.	Parallel to y-axis	$x = k$
5.	Parallel to $ax+by+c = 0$	$ax+by+k = 0$
6.	Perpendicular to $ax+by+c = 0$	$bx-ay+k = 0$
	Given	**Equation**
1.	Passing through the origin	$y = mx$
2.	Slope m, y-intercept c	$y = mx + c$
3.	Slope m, a point (x_1, y_1)	$y - y_1 = m(x - x_1)$
4.	Passing through two points (x_1, y_1), (x_2, y_2)	$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$
5.	x-intercept a , y-intercept b	$\frac{x}{a} + \frac{y}{b} = 1$

6

- Introduction
- Basic Proportionality Theorem
- Angle Bisector Theorem
- Similar Triangles
- Tangent chord theorem
- Pythagoras theorem

EUCLID
(300 BC)
Greece

Euclid's 'Elements' is one of the most influential works in the history of mathematics, serving as the main text book for teaching mathematics especially geometry.

Euclid's algorithm is an efficient method for computing the greatest common divisor.

GEOMETRY

There is geometry in the humming of the strings, there is music in the spacing of spheres - Pythagoras

6.1 Introduction

Geometry is a branch of mathematics that deals with the properties of various geometrical figures. The geometry which treats the properties and characteristics of various geometrical shapes with axioms or theorems, without the help of accurate measurements is known as theoretical geometry. The study of geometry improves one's power to think logically.

Euclid, who lived around 300 BC is considered to be the father of geometry. Euclid initiated a new way of thinking in the study of geometrical results by deductive reasoning based on previously proved results and some self evident specific assumptions called axioms or postulates.

Geometry holds a great deal of importance in fields such as engineering and architecture. For example, many bridges that play an important role in our lives make use of congruent and similar triangles. These triangles help to construct the bridge more stable and enables the bridge to withstand great amounts of stress and strain. In the construction of buildings, geometry can play two roles; one in making the structure more stable and the other in enhancing the beauty. Elegant use of geometric shapes can turn buildings and other structures such as the Taj Mahal into great landmarks admired by all. Geometric proofs play a vital role in the expansion and understanding of many branches of mathematics.

The basic proportionality theorem is attributed to the famous Greek mathematician Thales. This theorem is also called Thales theorem.

To understand the basic proportionality theorem, let us perform the following activity.

Activity

Draw any angle XAY and mark points (say five points) P_1, P_2, D, P_3 and B on arm AX such that $AP_1 = P_1P_2 = P_2D = DP_3 = P_3B = 1$ unit (say).

Through B draw any line intersecting arm AY at C. Again through D draw a line parallel to BC to intersect AC at E.

Now $AD = AP_1 + P_1P_2 + P_2D = 3$ units

and $DB = DP_3 + P_3B = 2$ units

$\therefore \quad \frac{AD}{DB} = \frac{3}{2}$

Measure AE and EC.

We observe that $\frac{AE}{EC} = \frac{3}{2}$

Thus, in ΔABC if $DE \parallel BC$, then $\frac{AD}{DB} = \frac{AE}{EC}$

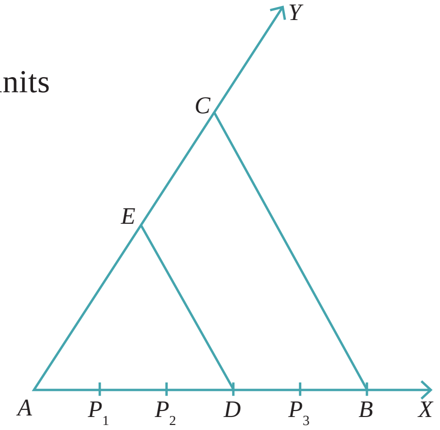


Fig. 6.1

We prove this result as a theorem known as Basic Proportionality Theorem or Thales Theorem as follows:

6.2 Basic proportionality and Angle Bisector theorems

Theorem 6.1 **Basic Proportionality theorem or Thales Theorem**

If a straight line is drawn parallel to one side of a triangle intersecting the other two sides, then it divides the two sides in the same ratio.

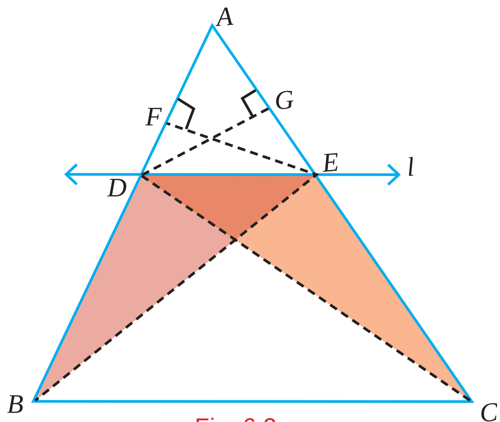


Fig. 6.2

Given: In a triangle ABC, a straight line l parallel to BC, intersects AB at D and AC at E.

To prove: $\frac{AD}{DB} = \frac{AE}{EC}$

Construction: Join BE, CD.

Draw $EF \perp AB$ and $DG \perp CA$.

Proof

Since, $EF \perp AB$, EF is the height of triangles ADE and DBE.

Area $(\Delta ADE) = \frac{1}{2} \times$ base $\times$ height $= \frac{1}{2} AD \times EF$ and

Area $(\Delta DBE) = \frac{1}{2} \times$ base $\times$ height $= \frac{1}{2} DB \times EF$

$$\therefore \qquad \frac{\text{area}(\Delta ADE)}{\text{area}(\Delta DBE)} = \frac{\frac{1}{2}AD \times EF}{\frac{1}{2}DB \times EF} = \frac{AD}{DB} \qquad (1)$$

Similarly, we get

$$\frac{\text{area}(\Delta ADE)}{\text{area}(\Delta DCE)} = \frac{\frac{1}{2} \times AE \times DG}{\frac{1}{2} \times EC \times DG} = \frac{AE}{EC} \qquad (2)$$

But, ΔDBE and ΔDCE are on the same base DE and between the same parallel straight lines BC and DE.

$$\therefore \qquad \text{area } (\Delta DBE) = \text{area } (\Delta DCE) \qquad (3)$$

Form (1), (2) and (3), we obtain $\frac{AD}{DB} = \frac{AE}{EC}$. Hence the theorem.

Corollary

If in a ΔABC, a straight line DE parallel to BC, intersects AB at D and AC at E, then

(i) $\frac{AB}{AD} = \frac{AC}{AE}$ (ii) $\frac{AB}{DB} = \frac{AC}{EC}$

Proof

(i) From Thales theorem, we have

$$\frac{AD}{DB} = \frac{AE}{EC}$$

$$\Longrightarrow \quad \frac{DB}{AD} = \frac{EC}{AE}$$

$$\Longrightarrow \quad 1 + \frac{DB}{AD} = 1 + \frac{EC}{AE}$$

$$\Longrightarrow \quad \frac{AD+DB}{AD} = \frac{AE+EC}{AE}$$

Thus, $\frac{AB}{AD} = \frac{AC}{AE}$

(ii) Similarly, we can prove

$$\frac{AB}{DB} = \frac{AC}{EC}$$

Do you know?

If $\frac{a}{b} = \frac{c}{d}$ then $\frac{a+b}{b} = \frac{c+d}{d}$.

This is called componendo rule.

Here, $\frac{DB}{AD} = \frac{EC}{AE}$

$$\Rightarrow \frac{AD+DB}{AD} = \frac{AE+EC}{AE}$$

by componendo rule.

Is the converse of this theorem also true? To examine this let us perform the following activity.

Activity

Draw an angle $\angle XAY$ and on the ray AX, mark points P_1, P_2, P_3, P_4 and B such that

$AP_1 = P_1P_2 = P_2P_3 = P_3P_4 = P_4B = 1$ unit (say).

Similarly, on ray AY, mark points Q_1, Q_2, Q_3, Q_4 and C such that

$AQ_1 = Q_1Q_2 = Q_2Q_3 = Q_3Q_4 = Q_4C = 2$ units (say).

Now join P_1Q_1 and BC.

Then $\frac{AP_1}{P_1B} = \frac{1}{4}$ and $\frac{AQ_1}{Q_1C} = \frac{2}{8} = \frac{1}{4}$

Thus, $\frac{AP_1}{P_1B} = \frac{AQ_1}{Q_1C}$

We observe that the lines P_1Q_1 and BC are parallel to each other. i.e., $P_1Q_1 \parallel BC$ (1)

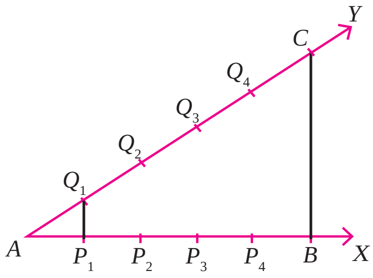


Fig. 6.3

Similarly, by joining P_2Q_2, P_3Q_3 and P_4Q_4 we see that

$$\frac{AP_2}{P_2B} = \frac{AQ_2}{Q_2C} = \frac{2}{3} \text{ and } P_2Q_2 \parallel BC \quad (2)$$

$$\frac{AP_3}{P_3B} = \frac{AQ_3}{Q_3C} = \frac{3}{2} \text{ and } P_3Q_3 \parallel BC \quad (3)$$

$$\frac{AP_4}{P_4B} = \frac{AQ_4}{Q_4C} = \frac{4}{1} \text{ and } P_4Q_4 \parallel BC \quad (4)$$

From (1), (2), (3) and (4) we observe that if a line divides two sides of a triangle in the same ratio, then the line is parallel to the third side.

In this direction, let us state and prove a theorem which is the converse of Thales theorem.

Theorem 6.2

Converse of Basic Proportionality Theorem (Converse of Thales Theorem)

If a straight line divides any two sides of a triangle in the same ratio, then the line must be parallel to the third side.

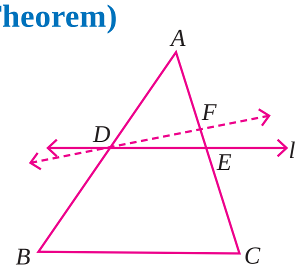


Fig. 6.4

Given: A line l intersects the sides AB and AC of ΔABC respectively at D and E such that $\frac{AD}{DB} = \frac{AE}{EC}$ (1)

To prove : $DE \parallel BC$

Construction : If DE is not parallel to BC, then draw a line $DF \parallel BC$.

Proof Since $DF \parallel BC$, by Thales theorem we get,

$$\frac{AD}{DB} = \frac{AF}{FC} \quad (2)$$

From (1) and (2), we get $\frac{AF}{FC} = \frac{AE}{EC} \implies \frac{AF+FC}{FC} = \frac{AE+EC}{EC}$

$$\frac{AC}{FC} = \frac{AC}{EC} \quad \therefore\ FC = EC$$

This is possible only when F and E coincide. Thus, $DE \parallel BC$.

Theorem 6.3 Angle Bisector Theorem

The internal (external) bisector of an angle of a triangle divides the opposite side internally (externally) in the ratio of the corresponding sides containing the angle.

Case (i) (Internally)

Given : In $\triangle ABC$, AD is the internal bisector of $\angle BAC$ which meets BC at D.

To prove : $\dfrac{BD}{DC} = \dfrac{AB}{AC}$

Construction : Draw $CE \parallel DA$ to meet BA produced at E.

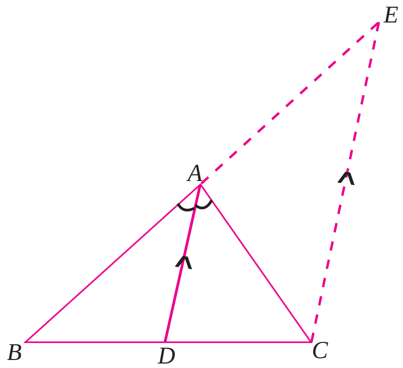


Fig. 6.5

Proof

Since $CE \parallel DA$ and AC is the transversal, we have

$\angle DAC = \angle ACE$ (alternate angles) (1)

and $\angle BAD = \angle AEC$ (corresponding angles) (2)

Since AD is the angle bisector of $\angle A$, $\angle BAD = \angle DAC$ (3)

From (1), (2) and (3), we have $\angle ACE = \angle AEC$

Thus in $\triangle ACE$, we have $AE = AC$ (sides opposite to equal angles are equal)

Now in $\triangle BCE$ we have, $CE \parallel DA$

$$\frac{BD}{DC} = \frac{BA}{AE} \qquad \text{(Thales theorem)}$$

$$\Longrightarrow \quad \frac{BD}{DC} = \frac{AB}{AC} \qquad (AE = AC)$$

Hence the theorem.

Case (ii) Externally ***(this part is not for examination)***

Given: In $\triangle ABC$, AD is the external bisector of $\angle BAC$ and intersects BC produced at D.

To prove: $\dfrac{BD}{DC} = \dfrac{AB}{AC}$

Construction: Draw $CE \parallel DA$ meeting AB at E.

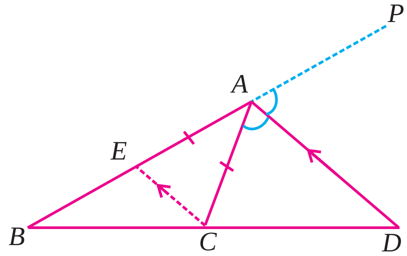


Fig. 6.6

Proof $CE \parallel DA$ and AC is a transversal,

$\angle ECA = \angle CAD$ (alternate angles) (1)

Also $CE \parallel DA$ and BP is a transversal

$$\angle CEA = \angle DAP \qquad \text{(corresponding angles)} \qquad (2)$$

But AD is the bisector of $\angle CAP$

$$\angle CAD = \angle DAP \qquad (3)$$

From (1), (2) and (3), we have

$$\angle CEA = \angle ECA$$

Thus, in $\triangle ECA$, we have $AC = AE$ (sides opposite to equal angles are equal)

In $\triangle BDA$, we have $EC \parallel AD$

$$\therefore \quad \frac{BD}{DC} = \frac{BA}{AE} \qquad \text{(Thales theorem)}$$

$$\Longrightarrow \quad \frac{BD}{DC} = \frac{BA}{AC} \qquad (AE = AC)$$

Hence the theorem.

Theorem 6.4 Converse of Angle Bisector Theorem

If a straight line through one vertex of a triangle divides the opposite side internally (externally) in the ratio of the other two sides, then the line bisects the angle internally (externally) at the vertex.

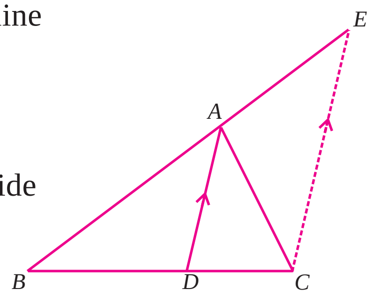


Fig. 6.7

Case (i) : (Internally)

Given : In $\triangle ABC$, the line AD divides the opposite side BC internally such that

$$\frac{BD}{DC} = \frac{AB}{AC} \qquad (1)$$

To prove : AD is the internal bisector of $\angle BAC$.

i.e., to prove $\angle BAD = \angle DAC$.

Construction :

Through C draw $CE \parallel AD$ meeting BA produced at E.

Proof Since $CE \parallel AD$, by Thales theorem, we have $\frac{BD}{DC} = \frac{BA}{AE}$ (2)

Thus, from (1) and (2) we have, $\frac{AB}{AE} = \frac{AB}{AC}$

$$\therefore \quad AE = AC$$

Now, in $\triangle ACE$, we have $\angle ACE = \angle AEC \qquad (AE = AC) \qquad (3)$

Since AC is a transversal of the parallel lines AD and CE,

we get, $\angle DAC = \angle ACE$ (alternate interior angles are equal) (4)

Also BE is a transversal of the parallel lines AD and CE.

we get $\angle BAD = \angle AEC$ (corresponding angles are equal) (5)

From (3), (4) and (5), we get

$$\angle BAD = \angle DAC$$

$\therefore$ AD is the angle bisector of $\angle BAC$.

Hence the theorem.

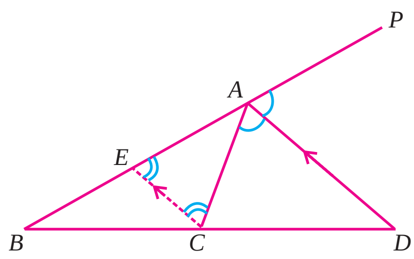


Fig. 6.8

Case (ii) Externally (this part is not for examination)

Given : In $\triangle ABC$, the line AD divides externally the opposite side BC produced at D.

such that $\dfrac{BD}{DC} = \dfrac{AB}{AC}$ (1)

To prove : AD is the bisector of $\angle PAC$,

i.e., to prove $\angle PAD = \angle DAC$

Construction : Through C draw $CE \parallel DA$ meeting BA at E.

Proof Since $CE \parallel DA$, by Thales theorem $\dfrac{BD}{DC} = \dfrac{BA}{EA}$ (2)

From (1) and (2), we have

$$\frac{AB}{AE} = \frac{AB}{AC} \quad \therefore\ AE = AC$$

In $\triangle ACE$, we have $\angle ACE = \angle AEC$ $(AE = AC)$ (3)

Since AC is a transversal of the parallel lines AD and CE, we have

$\angle ACE = \angle DAC$ (alternate interior angles) (4)

Also, BA is a transversal of the parallel lines AD and CE,

$\angle PAD = \angle AEC$ (corresponding angles) (5)

From (3) , (4) and (5), we get

$$\angle PAD = \angle DAC$$

$\therefore$ AD is the bisector of $\angle PAC$. Thus AD is the external bisector of $\angle BAC$

Hence the theorem.

Example 6.1

In $\triangle ABC$, $DE \parallel BC$ and $\frac{AD}{DB} = \frac{2}{3}$. If $AE = 3.7$cm, find EC.

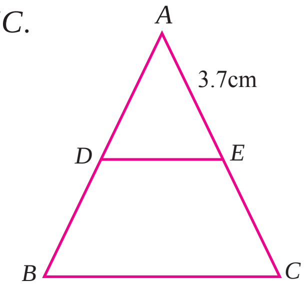


Fig. 6.9

Solution In $\triangle ABC$, $DE \parallel BC$

$$\therefore \quad \frac{AD}{DB} = \frac{AE}{EC} \qquad \text{(Thales theorem)}$$

$$\Longrightarrow \quad EC = \frac{AE \times DB}{AD}$$

Thus, $$EC = \frac{3.7 \times 3}{2} = 5.55\,\text{cm}$$

Example 6.2

In $\triangle PQR$, given that S is a point on PQ such that $ST \parallel QR$ and $\frac{PS}{SQ} = \frac{3}{5}$. If $PR = 5.6$cm, then find PT.

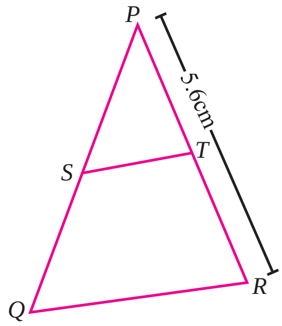


Fig. 6.10

Solution In $\triangle PQR$, we have $ST \parallel QR$ and by Thales theorem,

$$\frac{PS}{SQ} = \frac{PT}{TR} \qquad (1)$$

Let $PT = x$. Thus, $TR = PR - PT = 5.6 - x$.

From (1), we get $PT = TR\left(\frac{PS}{SQ}\right)$

$$x = (5.6 - x)\left(\frac{3}{5}\right)$$

$$5x = 16.8 - 3x$$

Thus, $$x = \frac{16.8}{8} = 2.1$$ That is, PT = 2.1cm.

Example 6.3

In a $\triangle ABC$, D and E are points on AB and AC respectively such that $\frac{AD}{DB} = \frac{AE}{EC}$ and $\angle ADE = \angle DEA$. Prove that $\triangle ABC$ is isosceles.

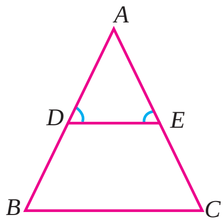


Fig. 6.11

Solution Since $\frac{AD}{DB} = \frac{AE}{EC}$, by converse of Thales theorem, $DE \parallel BC$

$$\therefore \quad \angle ADE = \angle ABC \text{ and} \qquad (1)$$

$$\angle DEA = \angle BCA \qquad (2)$$

But, given that $\angle ADE = \angle DEA \qquad (3)$

From (1), (2) and (3), we get $\angle ABC = \angle BCA$

$\therefore \quad AC = AB$ (If opposite angles are equal, then opposite sides are equal).

Thus, $\triangle ABC$ is isosceles.

Example 6.4

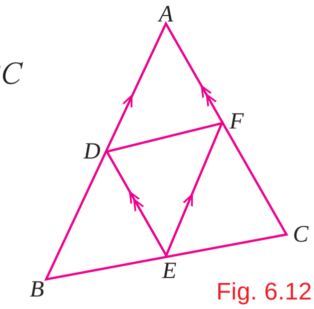


Fig. 6.12

The points D, E and F are taken on the sides AB, BC and CA *of* a $\triangle ABC$ respectively, such that $DE \parallel AC$ and $FE \parallel AB$.

Prove that $\dfrac{AB}{AD} = \dfrac{AC}{FC}$

Solution Given that in $\triangle ABC$, $DE \parallel AC$.

$$\therefore \quad \frac{BD}{DA} = \frac{BE}{EC} \quad \text{(Thales theorem)} \qquad (1)$$

Also, given that $FE \parallel AB$.

$$\therefore \quad \frac{BE}{EC} = \frac{AF}{FC} \quad \text{(Thales theorem)} \qquad (2)$$

From (1) and (2), we get

$$\frac{BD}{AD} = \frac{AF}{FC}$$

$$\Longrightarrow \quad \frac{BD+AD}{AD} = \frac{AF+FC}{FC} \quad \text{(componendo rule)}$$

Thus, $\dfrac{AB}{AD} = \dfrac{AC}{FC}$.

Example 6.5

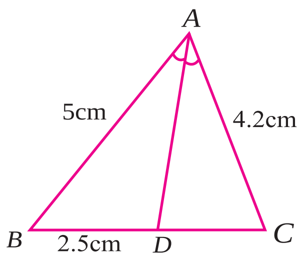


Fig. 6.13

In $\triangle ABC$, the internal bisector AD of $\angle A$ meets the side BC at D. If $BD = 2.5\,\text{cm}$, $AB = 5\,\text{cm}$ and $AC = 4.2\,\text{cm}$, then find DC.

Solution In $\triangle ABC$, AD is the internal bisector of $\angle A$.

$$\therefore \quad \frac{BD}{DC} = \frac{AB}{AC} \quad \text{(angle bisector theorem)}$$

$$\Longrightarrow \quad DC = \frac{BD \times AC}{AB}$$

Thus, $DC = \dfrac{2.5 \times 4.2}{5} = 2.1\,\text{cm}$.

Example 6.6

In $\triangle ABC$, AE is the external bisector of $\angle A$, meeting BC produced at E. If $AB = 10\,\text{cm}$, $AC = 6\,\text{cm}$ and $BC = 12\,\text{cm}$, then find CE.

Solution In $\triangle ABC$, AE is the external bisector of $\angle A$ meeting BC produced at E.

Let $CE = x$ cm. Now, by the angle bisector theorem, we have

$$\frac{BE}{CE} = \frac{AB}{AC} \Longrightarrow \frac{12+x}{x} = \frac{10}{6}$$

$$\Longrightarrow 3(12+x) = 5x.$$

$$\Longrightarrow x = 18.$$

Hence, $CE = 18\,\text{cm}$.

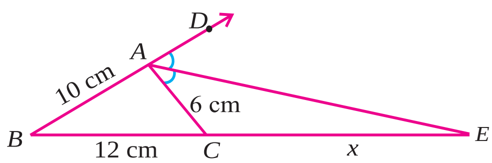


Fig. 6.14

Example 6.7

D is the midpoint of the side BC of $\triangle ABC$. If P and Q are points on AB and on AC such that DP bisects $\angle BDA$ and DQ bisects $\angle ADC$, then prove that $PQ \parallel BC$.

Solution In $\triangle ABD$, DP is the angle bisector of $\angle BDA$.

$\therefore$ $\quad \dfrac{AP}{PB} = \dfrac{AD}{BD}$ (angle bisector theorem) (1)

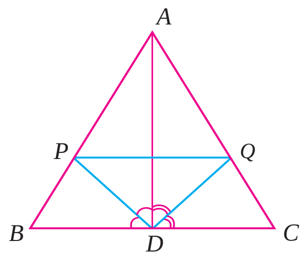


Fig. 6.15

In $\triangle ADC$, DQ is the bisector of $\angle ADC$

$\therefore$ $\quad \dfrac{AQ}{QC} = \dfrac{AD}{DC}$ (angle bisector theorem) (2)

But, $\quad BD = DC$ (D is the midpoint of BC)

Now (2) $\implies \dfrac{AQ}{QC} = \dfrac{AD}{BD}$ (3)

From (1) and (3) we get,

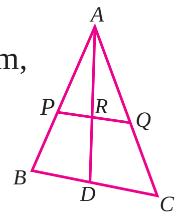
$$\frac{AP}{PB} = \frac{AQ}{QC}$$

Thus, $\quad PQ \parallel BC$. (converse of Thales theorem)

Exercise 6.1

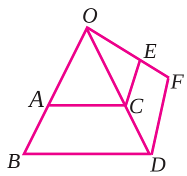
1. In a ΔABC, D and E are points on the sides AB and AC respectively such that $DE \parallel BC$.
   (i) If $AD = 6$ cm, $DB = 9$ cm and $AE = 8$ cm, then find AC.
   (ii) If $AD = 8$ cm, $AB = 12$ cm and $AE = 12$ cm, then find CE.
   (iii) If $AD = 4x-3$, $BD = 3x-1$, $AE = 8x-7$ and $EC = 5x-3$, then find the value of x.

2. In the figure, $AP = 3$ cm, $AR = 4.5$ cm, $AQ = 6$ cm, $AB = 5$ cm, and $AC = 10$ cm. Find the length of AD.



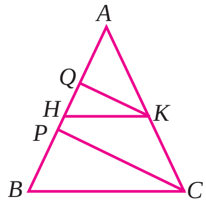
3. E and F are points on the sides PQ and PR respectively, of a $\triangle PQR$. For each of the following cases, verify $EF \parallel QR$.
   (i) $PE = 3.9$ cm, $EQ = 3$ cm, $PF = 3.6$ cm and $FR = 2.4$ cm.
   (ii) $PE = 4$ cm, $QE = 4.5$ cm, $PF = 8$ cm and $RF = 9$ cm.

4. In the figure,
   $AC \parallel BD$ and $CE \parallel DF$. If $OA = 12$cm, $AB = 9$ cm, $OC = 8$ cm and $EF = 4.5$ cm, then find FO.



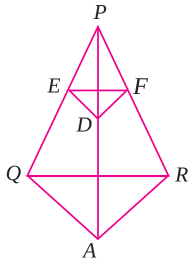
5. *ABCD* is a quadrilateral with *AB* parallel to *CD*. A line drawn parallel to *AB* meets *AD* at *P* and *BC* at *Q*. Prove that $\dfrac{AP}{PD} = \dfrac{BQ}{QC}$.

6. In the figure, $PC \parallel QK$ and $BC \parallel HK$. If *AQ* = 6 cm, *QH* = 4 cm, *HP* = 5 cm, *KC* = 18cm, then find *AK* and *PB*.



7. In the figure, $DE \parallel AQ$ and $DF \parallel AR$

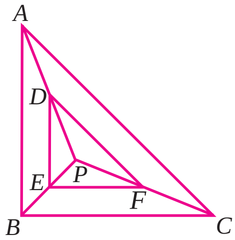
Prove that $EF \parallel QR$.



8. In the figure

$DE \parallel AB$ and $DF \parallel AC$.

Prove that $EF \parallel BC$.



9. In a $\triangle ABC$, *AD* is the internal bisector of $\angle A$, meeting *BC* at *D*.

(i) If *BD* = 2 cm, *AB* = 5 cm, *DC* = 3 cm find *AC*.

(ii) If *AB* = 5.6 cm, *AC* = 6 cm and *DC* = 3 cm find *BC*.

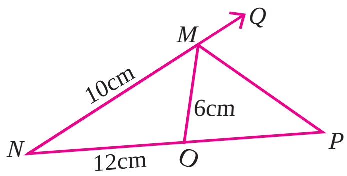
(iii) If *AB* = *x*, *AC* = *x*–2, *BD* = *x*+2 and *DC* = *x*–1 find the value of *x*.

10. Check whether *AD* is the bisector of $\angle A$ of $\triangle ABC$ in each of the following.

(i) *AB* = 4 cm, *AC* = 6 cm, *BD* = 1.6 cm, and *CD* = 2.4 cm.

(ii) *AB* = 6 cm, *AC* = 8cm, *BD* = 1.5cm and *CD* = 3cm.

11. In a $\triangle MNO$, *MP* is the external bisector of $\angle M$ meeting *NO* produced at *P*. If *MN* = 10 cm, *MO* = 6 cm, *NO* = 12 cm, then find *OP*.



12. In a quadrilateral *ABCD*, the bisectors of $\angle B$ and $\angle D$ intersect on *AC* at *E*. Prove that $\dfrac{AB}{BC} = \dfrac{AD}{DC}$.

13. The internal bisector of $\angle A$ of ΔABC meets *BC* at *D* and the external bisector of $\angle A$ meets *BC* produced at *E*. Prove that $\dfrac{BD}{BE} = \dfrac{CD}{CE}$.

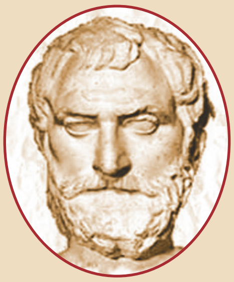
14. *ABCD* is a quadrilateral with *AB* =*AD*. If *AE* and *AF* are internal bisectors of $\angle BAC$ and $\angle DAC$ respectively, then prove that $EF \parallel BD$.

6.3 Similar triangles

In class VIII, we have studied congruence of triangles in detail. We have learnt that two geometrical figures are congruent if they have the same size and shape. In this section, we shall study about those geometrical figures which have the same shape but not necessarily the same size. Such geometrical figures are called similar.

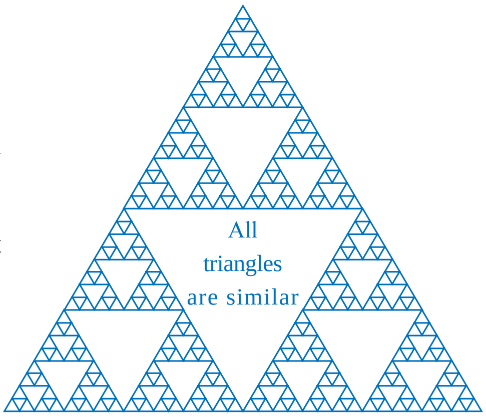
On looking around us, we see many objects which are of the same shape but of same or different sizes. For example, leaves of a tree have almost the same shapes but same or different sizes. Similarly photographs of different sizes developed from the same negative are of same shape but different sizes. All those objects which have the same shape but different sizes are called similar objects.

Thales said to have introduced Geometry in Greece, is believed to have found the heights of the Pyramids in Egypt, using shadows and the principle of similar triangles. Thus the use of similar triangles has made possible the measurements of heights and distances. He observed that the base angles of an isosceles triangle are equal. He used the idea of similar triangles and right triangles in practical geometry.



Thales of Miletus
(624-546 BC)
Greece

Thales was the first known philosopher, scientist and mathematician. He is credited with the first use of deductive reasoning applied to geometry. He discovered many prepositions in geometry. His method of attacking problems invited the attention of many mathematicians. He also predicted an eclipse of the Sun in 585 BC.



It is clear that the congruent figures are similar but the converse need not be true. In this section, we shall discuss only the similarity of triangles and apply this knowledge in solving problems. The following simple activity helps us to visualize similar triangles.

Activity

- Take a cardboard and make a triangular hole in it.
- Expose this cardboard to Sunlight at about one metre above the ground .
- Move it towards the ground to see the formation of a sequence of triangular shapes on the ground.
- Moving close to the ground, the image becomes smaller and smaller. Moving away from the ground, the image becomes larger and larger.
- You see that, the size of the angles forming the three vertices of the triangle would always be the same, even though their sizes are different.

Definition

Two triangles are similar if

(i) their corresponding angles are equal (or)

(ii) their corresponding sides have lengths in the same ratio (or proportional), which is equivalent to saying that one triangle is an enlargement of other.

Thus, two triangles $\triangle ABC$ and $\triangle DEF$ are similar if

(i) $\angle A = \angle D,\ \angle B = \angle E,\ \angle C = \angle F$ (or)

(ii) $\dfrac{AB}{DE} = \dfrac{BC}{EF} = \dfrac{CA}{FD}$.

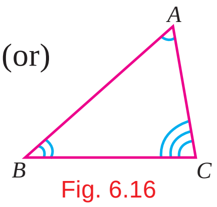


Fig. 6.16

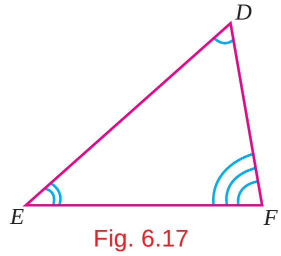


Fig. 6.17

Here, the vertices A, B and C correspond to the vertices D, E and F respectively. Symbolically, we write the similarity of these two triangles as $\triangle ABC \sim \triangle DEF$ and read it as $\triangle ABC$ is similar to $\triangle DEF$. The symbol '$\sim$' stands for 'is similar to'.

Remarks

Similarity of $\triangle ABC$ and $\triangle DEF$ can also be expressed symbolically using correct correspondence of their vertices as $\triangle BCA \sim \triangle EFD$ and $\Delta CAB \sim \Delta FDE$.

6.3.1 Criteria for similarity of triangles

The following three criteria are sufficient to prove that two triangles are similar.

(i) AA(Angle-Angle) similarity criterion

If two angles of one triangle are respectively equal to two angles of another triangle, then the two triangles are similar.

Remark If two angles of a triangle are respectively equal to two angles of another triangle then their third angles will also be equal. Therefore AA similarity criterion is also referred to AAA criteria.

(ii) SSS (Side-Side-Side) similarity criterion for Two Triangles

In two triangles, if the sides of one triangle are proportional (in the same ratio) to the sides of the other triangle, then their corresponding angles are equal and hence the two triangles are similar.

(iii) SAS (Side-Angle-Side) similarity criterion for Two Triangles

If one angle of a triangle is equal to one angle of the other triangle and if the corresponding sides including these angles are proportional, then the two triangles are similar.

Let us list out a few results without proofs on similarity of triangles.

(i) The ratio of the areas of two similar triangles is equal to the ratio of the squares of their corresponding sides.

(ii) If a perpendicular is drawn from the vertex of a right angled triangle to its hypotenuse, then the triangles on each side of the perpendicular are similar to the whole triangle.

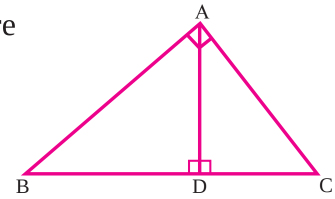


Fig. 6.18

Here, (a) $\Delta\ DBA \sim \Delta\ ABC$

(b) $\Delta\ DAC \sim \Delta\ ABC$

(c) $\Delta\ DBA \sim \Delta\ DAC$

(iii) If two triangles are similar, then the ratio of the corresponding sides is equal to the ratio of their corresponding altitudes.

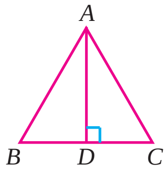


Fig. 6.19

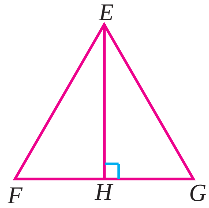


Fig. 6.20

i.e., if $\Delta ABC \sim \Delta EFG$, then $\dfrac{AB}{EF} = \dfrac{BC}{FG} = \dfrac{CA}{GE} = \dfrac{AD}{EH}$

(iv) If two triangles are similar, then the ratio of the corresponding sides is equal to the ratio of the corresponding perimeters.

If, $\Delta\ ABC \sim \Delta\ DEF$, then $\dfrac{AB}{DE} = \dfrac{BC}{EF} = \dfrac{CA}{FD} = \dfrac{AB+BC+CA}{DE+EF+FD}$.

Example 6.8

In ΔPQR, $AB \parallel QR$. If AB is 3 cm, PB is 2cm and PR is 6 cm, then find the length of QR.

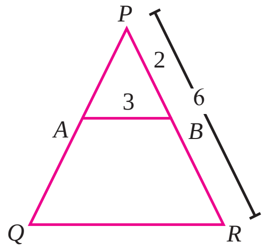


Fig. 6.21

Solution Given AB is 3cm, PB is 2 cm PR is 6 cm and $AB \parallel QR$

In ΔPAB and ΔPQR

$\angle PAB = \angle PQR$ (corresponding angles)

and $\angle P$ is common.

$\therefore\ \Delta PAB \sim \Delta PQR$ (AA similarity criterion)

Since corresponding sides are proportional,

$$\frac{AB}{QR} = \frac{PB}{PR}$$

$$QR = \frac{AB \times PR}{PB}$$

$$= \frac{3 \times 6}{2}$$

Thus, $QR = 9$ cm.

Example 6.9

A man of height 1.8 m is standing near a Pyramid. If the shadow of the man is of length 2.7 m and the shadow of the Pyramid is 210 m long at that instant, find the height of the Pyramid.

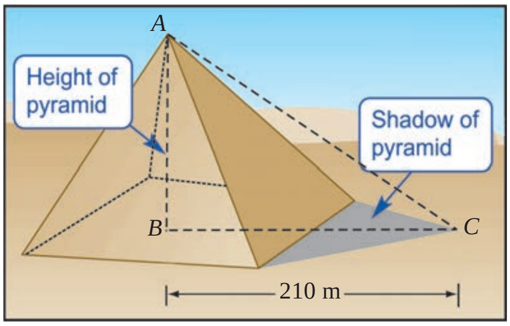


Fig. 6.22

Solution Let AB and DE be the heights of the Pyramid and the man respectively.

Let BC and EF be the lengths of the shadows of the Pyramid and the man respectively.

In ΔABC and ΔDEF, we have

$$\angle ABC = \angle DEF = 90^{\circ}$$
$$\angle BCA = \angle EFD$$

(angular elevation is same at the same instant)

$$\therefore \Delta ABC \sim \Delta DEF \qquad \text{(AA similarity criterion)}$$

Thus, $$\frac{AB}{DE} = \frac{BC}{EF}$$

$$\Longrightarrow \frac{AB}{1.8} = \frac{210}{2.7} \Longrightarrow AB = \frac{210}{2.7} \times 1.8 = 140.$$

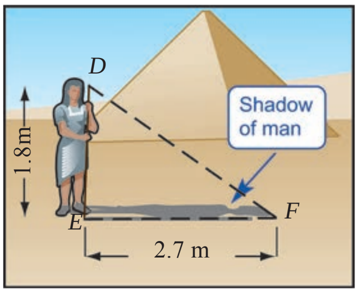


Fig. 6.23

Hence, the height of the Pyramid is 140 m.

Example 6.10

A man sees the top of a tower in a mirror which is at a distance of 87.6 m from the tower. The mirror is on the ground, facing upward. The man is 0.4 m away from the mirror, and the distance of his eye level from the ground is 1.5 m. How tall is the tower? (The foot of man, the mirror and the foot of the tower lie along a straight line).

Solution Let AB and ED be the heights of the man and the tower respectively. Let C be the point of incidence of the tower in the mirror.

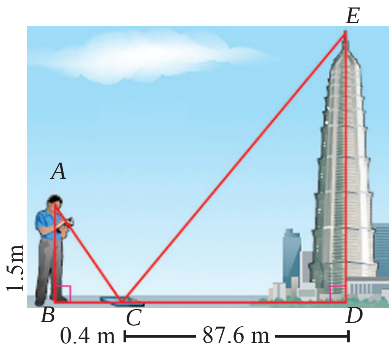


Fig. 6.24

In ΔABC and ΔEDC, we have

$$\angle ABC = \angle EDC = 90^{\circ}$$
$$\angle BCA = \angle DCE$$

(angular elevation is same at the same instant. i.e., the angle of incidence and the angle of reflection are same.)

$$\therefore \qquad \Delta ABC \sim \Delta EDC \qquad \text{(AA similarity criterion)}$$

Thus, $$\frac{ED}{AB} = \frac{DC}{BC} \qquad \text{(corresponding sides are proportional)}$$

$$ED = \frac{DC}{BC} \times AB = \frac{87.6}{0.4} \times 1.5 = 328.5$$

Hence, the height of the tower is 328.5 m.

Example 6.11

The image of a tree on the film of a camera is of length 35 mm, the distance from the lens to the film is 42 mm and the distance from the lens to the tree is 6 m. How tall is the portion of the tree being photographed?

Solution Let AB and EF be the heights of the portion of the tree and its image on the film respectively.

Let the point C denote the lens.

Let CG and CH be altitudes of ΔACB and ΔFEC.

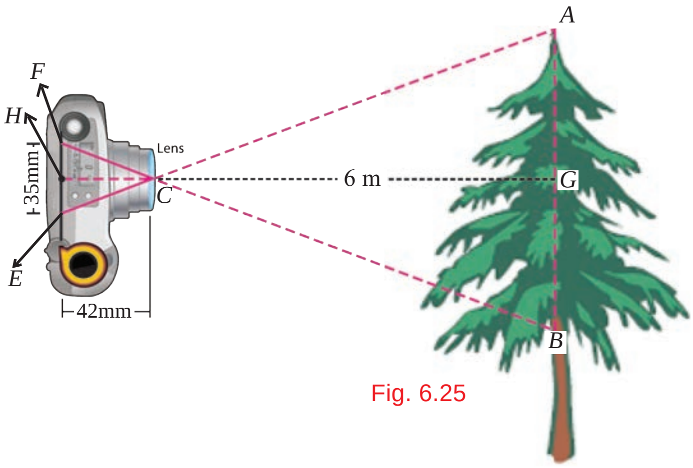


Fig. 6.25

Clearly, AB || FE.

In ΔACB and ΔFEC,

$\angle BAC = \angle FEC$

$\angle ECF = \angle ACB$ (vertically opposite angles)

$\therefore \ \Delta ACB \sim \Delta ECF$ (AA criterion)

Thus, $$\frac{AB}{EF} = \frac{CG}{CH}$$

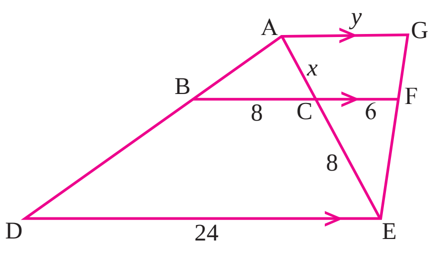
$$\Longrightarrow \quad AB = \frac{CG}{CH} \times EF = \frac{6 \times 0.035}{0.042} = 5.$$

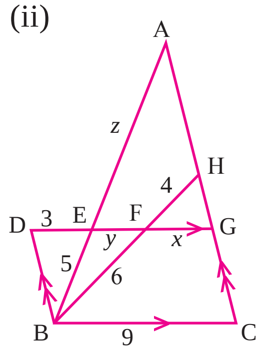
Hence, the height of the tree photographed is 5m.

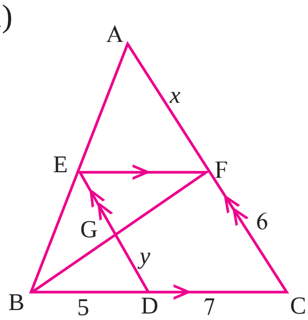
Exercise 6.2

1. Find the unknown values in each of the following figures. All lengths are given in centimetres. (All measures are not in scale)

(i) (ii) (iii)



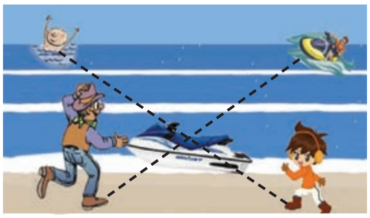




2. The image of a man of height 1.8 m, is of length 1.5 cm on the film of a camera. If the film is 3 cm from the lens of the camera, how far is the man from the camera?

3. A girl of height 120 cm is walking away from the base of a lamp-post at a speed of 0.6 m/sec. If the lamp is 3.6 m above the ground level, then find the length of her shadow after 4 seconds.

4. A girl is in the beach with her father. She spots a swimmer drowning. She shouts to her father who is 50 m due west of her. Her father is 10 m nearer to a boat than the girl. If her father uses the boat to reach the swimmer, he has to travel a distance 126 m from that boat. At the same time, the girl spots a man riding a water craft who is 98 m away from the boat. The man on the water craft is due east of the swimmer. How far must the man travel to rescue the swimmer? (Hint : see figure). (Not for the examination)



5. P and Q are points on sides AB and AC respectively, of ΔABC. If $AP = 3$ cm, $PB = 6$ cm, $AQ = 5$ cm and $QC = 10$ cm, show that $BC = 3\,PQ$.

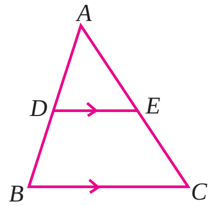
6. In ΔABC, $AB = AC$ and $BC = 6$ cm. D is a point on the side AC such that $AD = 5$ cm and $CD = 4$ cm. Show that $\Delta BCD \sim \Delta ACB$ and hence find BD.

7. The points D and E are on the sides AB and AC *of* ΔABC respectively, such that $DE \parallel BC$. If $AB = 3\,AD$ and the area of ΔABC is 72 cm^2, then find the area of the quadrilateral $DBCE$.

8. The lengths of three sides of a triangle ABC are 6 cm, 4 cm and 9 cm. $\triangle PQR \sim \triangle ABC$. One of the lengths of sides of $\triangle PQR$ is 35cm. What is the greatest perimeter possible for $\triangle PQR$?

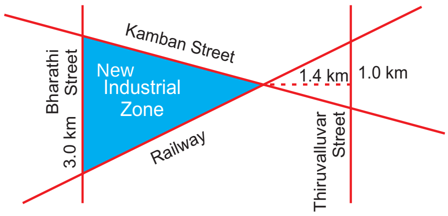
9. In the figure, $DE \parallel BC$ and $\frac{AD}{BD} = \frac{3}{5}$, calculate the value of

(i) $\dfrac{\text{area of } \Delta ADE}{\text{area of } \Delta ABC}$, (ii) $\dfrac{\text{area of trapezium } BCED}{\text{area of } \Delta ABC}$

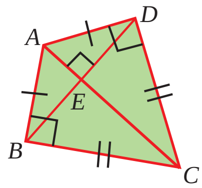


10. The government plans to develop a new industrial zone in an unused portion of land in a city.

The shaded portion of the map shown on the right, indicates the area of the new industrial zone. Find the area of the new industrial zone.

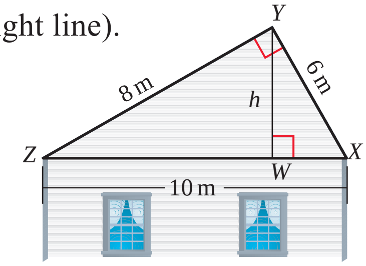


11. A boy is designing a diamond shaped kite, as shown in the figure where $AE = 16$ cm, $EC = 81$ cm. He wants to use a straight cross bar BD. How long should it be?



12. A student wants to determine the height of a flagpole. He placed a small mirror on the ground so that he can see the reflection of the top of the flagpole. The distance of the mirror from him is 0.5 m and the distance of the flagpole from the mirror is 3 m. If his eyes are 1.5 m above the ground level, then find the height of the flagpole.(The foot of student, mirror and the foot of flagpole lie along a straight line).

13. A roof has a cross section as shown in the diagram,
(i) Identify the similar triangles
(ii) Find the height h of the roof.



Theorem 6.5 **Pythagoras theorem (Baudhayan theorem)**

In a right angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.

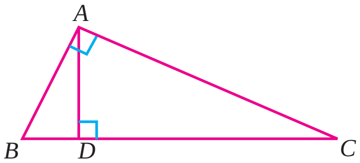


Fig. 6.26

Given : In a right angled $\triangle ABC$, $\angle A = 90^\circ$.

To prove : $BC^2 = AB^2 + AC^2$

Construction : Draw $AD \perp BC$

Proof

In triangles ABC and DBA, $\angle B$ is the common angle.

Also, we have $\angle BAC = \angle ADB = 90^\circ$.

$\therefore \triangle ABC \sim \triangle DBA$ (AA similarity criterion)

Thus, their corresponding sides are proportional.

Hence, $\dfrac{AB}{DB} = \dfrac{BC}{BA}$

$\therefore \quad AB^2 = DB \times BC$ (1)

Similarly, we have $\triangle ABC \sim \triangle DAC$.

Thus, $\dfrac{BC}{AC} = \dfrac{AC}{DC}$

$\therefore \quad AC^2 = BC \times DC$ (2)

Adding (1) and (2) we get,

$$AB^2 + AC^2 = BD \times BC + BC \times DC$$
$$= BC(BD + DC)$$
$$= BC \times BC = BC^2$$

Thus, $BC^2 = AB^2 + AC^2$. Hence the Pythagoras theorem.

Remarks

The Pythagoras theorem has two fundamental aspects; one is about areas and the other is about lengths. Hence this landmark thorem connects Geometry and Algebra.The converse of Pythagoras theorem is also true. It was first mentioned and proved by Euclid.

The statement is given below. (Proof is left as an exercise.)

Theorem 6.6 **Converse of Pythagoras theorem**

In a triangle, if the square of one side is equal to the sum of the squares of the other two sides, then the angle opposite to the first side is a right angle.

6.4 Circles and Tangents

A straight line associated with circles is a tangent line which touches the circle at just one point. In geometry, tangent lines to circles play an important role in many geometrical constructions and proofs. . In this section, let us state some results based on circles and tangents and prove an important theorem known as Tangent-Chord thorem. If we consider a straight line and a circle in a plane, then there are three possibilities- they may not intersect at all, they may intersect at two points or they may touch each other at exactly one point. Now look at the following figures.

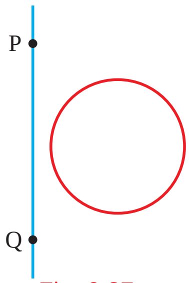


Fig. 6.27

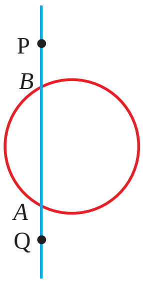


Fig. 6.28

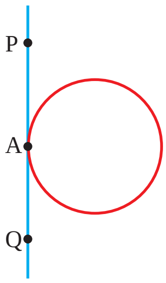


Fig. 6.29

In Fig. 6.27, the circle and the straight line PQ have no common point.

In Fig. 6.28, the straight line PQ cuts the circle at two distinct points A and B. In this case, PQ is called a secant to the circle.

In Fig. 6.29, the straight line PQ and the circle have exactly one common point. Equivalently the straight line touches the circle at only one point. The straight line PQ is called the tangent to the circle at A.

Definition

A straight line which touches a circle at only one point is called a tangent to the circle and the point at which it touches the circle is called its point of contact.

Theorems based on circles and tangents (without proofs)

1. A tangent at any point on a circle is perpendicular to the radius through the point of contact .
2. Only one tangent can be drawn at any point on a circle. However, from an exterior point of a circle two tangents can be drawn to the circle.
3. The lengths of the two tangents drawn from an exterior point to a circle are equal.
4. If two circles touch each other, then the point of contact of the circles lies on the line joining the centres.
5. If two circles touch externally, the distance between their centres is equal to the sum of their radii.
6. If two circles touch internally, the distance between their centres is equal to the difference of their radii.

Theorem 6.7 Tangent-Chord theorem

If from the point of contact of tangent (of a circle), a chord is drawn, then the angles which the chord makes with the tangent line are equal respectively to the angles formed by the chord in the corresponding alternate segments.

Given : O is the centre of the circle. ST is the tangent at A, and AB is a chord. P and Q are any two points on the circle in the opposite sides of the chord AB.

To prove : (i) $\angle BAT = \angle BPA$ (ii) $\angle BAS = \angle AQB$.

Construction: Draw the diameter AC of the circle. Join B and C.

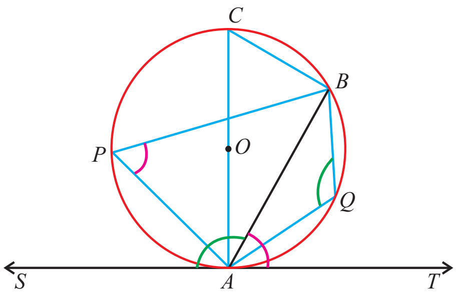


Fig. 6.30

Proof

Statement	Reason
$\angle ABC = 90^\circ$	angle in a semi-circle is 90°
$\angle CAB + \angle BCA = 90^\circ$	sum of two acute angles of a right $\triangle ABC$. (1)
$\angle CAT = 90^\circ$	diameter is $\perp$ to the tangent at the point of contact.
$\Longrightarrow \angle CAB + \angle BAT = 90^\circ$	(2)
$\angle CAB + \angle BCA = \angle CAB + \angle BAT$	from (1) and (2)
$\Longrightarrow \angle BCA = \angle BAT$	(3)

$\angle BCA = \angle BPA$ angles in the same segment (4)

$\angle BAT = \angle BPA$. Hence (i). from (3) and (4) (5)

Now $\angle BPA + \angle AQB = 180^\circ$ opposite angles of a cyclic quadrilateral

$\Longrightarrow \angle BAT + \angle AQB = 180^\circ$ from (5) (6)

Also $\angle BAT + \angle BAS = 180^\circ$ linear pair (7)

$\angle BAT + \angle AQB = \angle BAT + \angle BAS$ from (6) and (7)

$\angle BAS = \angle AQB$. Hence (ii).

Thus, the Tangent -Chord theorem is proved.

Theorem 6.8 Converse of Tangent-Chord theorem

If in a circle, through one end of a chord, a straight line is drawn making an angle equal to the angle in the alternate segment, then the straight line is a tangent to the circle.

Definition

Let P be a point on a line segment AB. The product $PA \times PB$ represents the area of the rectangle whose sides are PA and PB.

This product is called the area of the rectangle contained by the parts PA and PB of the line segment AB.

Theorem 6.9

If two chords of a circle intersect either inside or outside of the circle, then the area of the rectangle contained by the segments of one chord is equal to the area of the rectangle contained by the segments of the other chord.

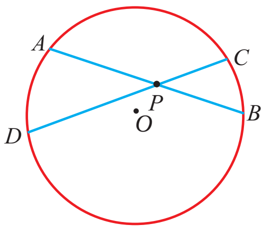


Fig. 6.31

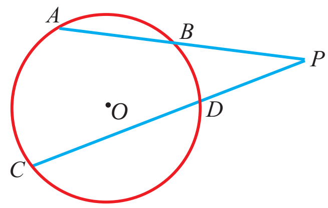


Fig. 6.32

In Fig.6.31, two chords AB and CD intersect at P inside the circle with centre at O. Then $PA \times PB = PC \times PD$. In Fig. 6.32, the chords AB and CD intersect at P outside the circle with centre O. Then $PA \times PB = PC \times PD$.

Example 6.12

Let PQ be a tangent to a circle at A and AB be a chord. Let C be a point on the circle such that $\angle BAC = 54^\circ$ and $\angle BAQ = 62^\circ$. Find $\angle ABC$..

Solution Since PQ is a tangent at A and AB is a chord, we have

$\angle BAQ = \angle ACB = 62^\circ$. (tangent-chord theorem)

Also, $\angle BAC + \angle ACB + \angle ABC = 180^\circ$.

(sum of all angles in a triangle is 180°)

Thus, $\angle ABC = 180^\circ - (\angle BAC + \angle ACB)$

Hence, $\angle ABC = 180^\circ - (54^\circ + 62^\circ) = 64^\circ$.

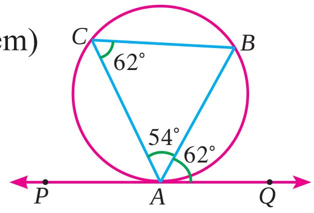


Fig. 6.33

Example 6.13

Find the value of x in each of the following diagrams.

(i)

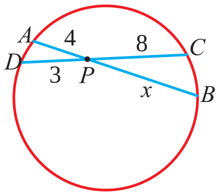


Fig. 6.34

(ii)

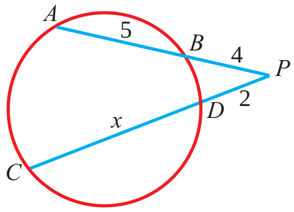


Fig. 6.35

Solution (i) We have $PA \,.\, PB = PC \,.\, PD$

$$PB = \frac{PC.PD}{PA}$$

Thus, $x = \dfrac{8 \times 3}{4} = 6.$

(ii) We have $PC \,.\, PD = PA.\, PB$

$$(2+x)\, 2 = 9 \times 4$$

$x + 2 = 18$. Thus, $x = 16$.

Example 6.14

In the figure, tangents PA and PB are drawn to a circle with centre O from an external point P. If CD is a tangent to the circle at E and $AP = 15$ cm, find the perimeter of ΔPCD

Solution We know that the lengths of the two tangents from an exterior point to a circle are equal.

$\therefore$ $CA = CE$, $DB = DE$ and $PA = PB$.

Now, the perimeter of $\Delta PCD = PC + CD + DP$

$= PC + CE + ED + DP$

$= PC + CA + DB + DP$

$= PA + PB = 2\,PA$ $(PB = PA)$

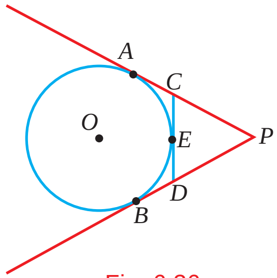


Fig. 6.36

Thus, the perimeter of $\Delta PCD = 2 \times 15 = 30$ cm.

Example 6.15

$ABCD$ is a quadrilateral such that all of its sides touch a circle. If $AB = 6$ cm, $BC = 6.5$ cm and $CD = 7$ cm , then find the length of AD.

Solution Let P, Q, R and S be the points where the circle touches the quadrilateral. We know that the lengths of the two tangents drawn from an exterior point to a circle are equal.

Thus, we have, $AP = AS$, $BP = BQ$, $CR = CQ$ and $DR = DS$.

Hence, $AP + BP + CR + DR = AS + BQ + CQ + DS$

$$\Longrightarrow \quad AB + CD = AD + BC.$$

$$\Longrightarrow \quad AD = AB + CD - BC$$

$$= 6 + 7 - 6.5 = 6.5$$

Thus, $AD = 6.5$ cm.

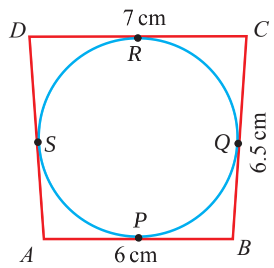
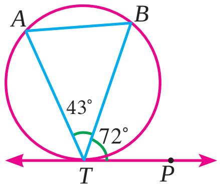


Fig. 6.37

Exercise 6.3



1. In the figure TP is a tangent to a circle. A and B are two points on the circle. If $\angle BTP = 72^\circ$ and $\angle ATB = 43^\circ$ find $\angle ABT$.

2. AB and CD are two chords of a circle which intersect each other internally at P. (i) If $CP = 4$ cm, $AP = 8$ cm, $PB = 2$ cm, then find PD.

   (ii) If $AP = 12$ cm, $AB = 15$ cm, $CP = PD$, then find CD

3. AB and CD are two chords of a circle which intersect each other externally at P

   (i) If $AB = 4$ cm $BP = 5$ cm and $PD = 3$ cm, then find CD.

   (ii) If $BP = 3$ cm, $CP = 6$ cm and $CD = 2$ cm, then find AB

4. A circle touches the side BC of ΔABC at P, AB and AC produced at Q and R respectively, prove that $AQ = AR = \frac{1}{2}$(perimeter of ΔABC)

5. If all sides of a parallelogram touch a circle, show that the parallelogram is a rhombus.

6. A lotus is 20 cm above the water surface in a pond and its stem is partly below the water surface. As the wind blew, the stem is pushed aside so that the lotus touched the water 40 cm away from the original position of the stem. How much of the stem was below the water surface originally?

7. A point O in the interior of a rectangle $ABCD$ is joined to each of the vertices A, B, C and D. Prove that $OA^2 + OC^2 = OB^2 + OD^2$

Exercise 6.4

Choose the correct answer

1. If a straight line intersects the sides AB and AC of a $\triangle ABC$ at D and E respectively and is parallel to BC, then $\frac{AE}{AC} =$

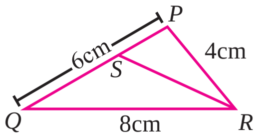
   (A) $\frac{AD}{DB}$ (B) $\frac{AD}{AB}$ (C) $\frac{DE}{BC}$ (D) $\frac{AD}{EC}$

2. In $\triangle ABC$, DE is $\parallel$ to BC, meeting AB and AC at D and E.

   If $AD = 3$ cm, $DB = 2$ cm and $AE = 2.7$ cm , then AC is equal to

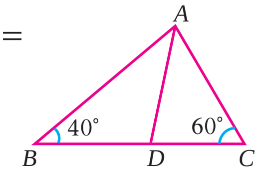
   (A) 6.5 cm (B) 4.5 cm (C) 3.5 cm (D) 5.5 cm

3. In $\triangle PQR$, RS is the bisector of $\angle R$. If $PQ = 6$ cm, $QR = 8$ cm, $RP = 4$ cm then PS is equal to



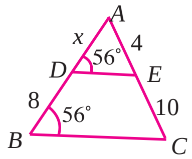
   (A) 2 cm (B) 4 cm (C) 3 cm (D) 6 cm

4. In figure, if $\frac{AB}{AC} = \frac{BD}{DC}$, $\angle B = 40^\circ$, and $\angle C = 60^\circ$, then $\angle BAD =$



   (A) 30° (B) 50° (C) 80° (D) 40°

5. In the figure, the value x is equal to



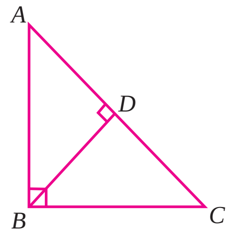
   (A) $4 \cdot 2$ (B) $3 \cdot 2$

   (C) $0 \cdot 8$ (D) $0 \cdot 4$

6. In triangles ABC and DEF, $\angle B = \angle E$, $\angle C = \angle F$, then

   (A) $\frac{AB}{DE} = \frac{CA}{EF}$ (B) $\frac{BC}{EF} = \frac{AB}{FD}$ (C) $\frac{AB}{DE} = \frac{BC}{EF}$ (D) $\frac{CA}{FD} = \frac{AB}{EF}$

7. From the given figure, identify the wrong statement.



   (A) $\Delta ADB \sim \Delta ABC$ (B) $\Delta ABD \sim \Delta ABC$

   (C) $\Delta BDC \sim \Delta ABC$ (D) $\Delta ADB \sim \Delta BDC$

8. If a vertical stick 12 m long casts a shadow 8 m long on the ground and at the same time a tower casts a shadow 40 m long on the ground, then the height of the tower is

   (A) 40 m (B) 50 m (C) 75 m (D) 60 m

9. The sides of two similar triangles are in the ratio 2:3, then their areas are in the ratio

   (A) 9:4 (B) 4:9 (C) 2:3 (D) 3:2

10. Triangles ABC and DEF are similar. If their areas are 100 cm^2 and 49 cm^2 respectively and BC is 8.2 cm then $EF =$

    (A) 5.47 cm (B) 5.74 cm (C) 6.47 cm (D) 6.74 cm

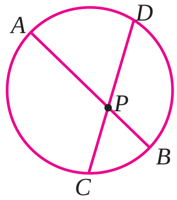
11. The perimeters of two similar triangles are 24 cm and 18 cm respectively. If one side of the first triangle is 8 cm, then the corresponding side of the other triangle is

    (A) 4 cm (B) 3 cm (C) 9 cm (D) 6 cm

12. AB and CD are two chords of a circle which when produced to meet at a point P such that $AB = 5$ cm, $AP = 8$ cm, and $CD = 2$ cm then $PD =$

(A) 12 cm (B) 5 cm (C) 6 cm (D) 4 cm

13. In the adjoining figure, chords AB and CD intersect at P. If $AB = 16$ cm, $PD = 8$ cm, $PC = 6$ and AP >PB, then AP =

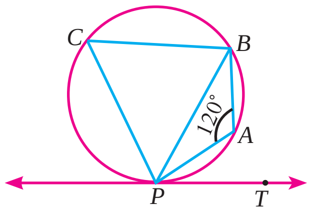


(A) 8 cm (B) 4 cm (C) 12 cm (D) 6 cm

14. A point P is 26 cm away from the centre O of a circle and PT is the tangent drawn from P to the circle is 10 cm, then OT is equal to

(A) 36 cm (B) 20 cm (C) 18 cm (D) 24 cm

15. In the figure, if $\angle PAB = 120^\circ$ then $\angle BPT =$

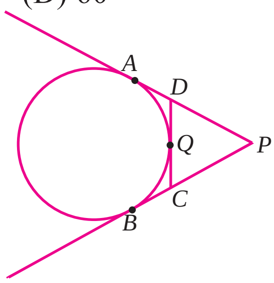


(A) 120° (B) 30° (C) 40° (D) 60°

16. If the tangents PA and PB from an external point P to circle with centre O are inclined to each other at an angle of 40°, then $\angle POA =$

(A) 70° (B) 80° (C) 50° (D) 60°

17. In the figure, PA and PB are tangents to the circle drawn from an external point P. Also CD is a tangent to the circle at Q. If $PA = 8$ cm and $CQ = 3$ cm, then PC is equal to



(A) 11 cm (B) 5 cm (C) 24 cm (D) 38 cm

18. ΔABC is a right angled triangle where $\angle B = 90^\circ$ and $BD \perp AC$. If BD = 8 cm, $AD = 4$ cm, then CD is

(A) 24 cm (B) 16 cm (C) 32 cm (D) 8 cm

19. The areas of two similar triangles are 16 cm^2 and 36 cm^2 respectively. If the altitude of the first triangle is 3 cm, then the corresponding altitude of the other triangle is

(A) 6.5 cm (B) 6 cm (C) 4 cm (D) 4.5 cm

20. The perimeter of two similar triangles ΔABC and ΔDEF are 36 cm and 24 cm respectively. If $DE = 10$ cm, then AB is

(A) 12 cm (B) 20 cm (C) 15 cm (D) 18 cm

7

- Introduction
- Identities
- Heights and Distances

Hipparchus
(190 - 120 B.C.)
Greece

Hipparchus developed trigonometry, constructed trigonometric tables and solved several problems of spherical trigonometry. With his solar and lunar theories and his trigonometry, he may have been the first to develop a reliable method to predict solar eclipses.

Hipparchus is credited with the invention or improvement of several astronomical instruments, which were used for a long time for naked-eye observations.

TRIGONOMETRY

There is perhaps nothing which so occupies the middle position of mathematics as trigonometry – J.F. Herbart

7.1 Introduction

Trigonometry was developed to express relationship between the sizes of arcs in circles and the chords determining those arcs. After 15th century it was used to relate the measure of angles in a triangle to the lengths of the sides of the triangle. The creator of Trigonometry is said to have been the Greek Hipparchus of the second century B.C. The word **Trigonometry** which means triangle measurement, is credited to Bartholomaus Pitiscus (1561-1613).

We have learnt in class IX about various trigonometric ratios, relation between them and how to use trigonometric tables in solving problems.

In this chapter, we shall learn about trigonometric identities, application of trigonometric ratios in finding heights and distances of hills, buildings etc., without actually measuring them.

7.2 Trigonometric identities

We know that an equation is called an identity when it is true for all values of the variable(s) for which the equation is meaningful. For example, the equation $(a+b)^2 = a^2 + 2ab + b^2$ is an identity since it is true for all real values of a and b.

Likewise, an equation involving trigonometric ratios of an angle is called a trigonometric identity, if it is true for all values of the angle(s) involved in the equation. For example, the equation $(\sin\theta + \cos\theta)^2 - (\sin\theta - \cos\theta)^2 = 4\sin\theta\cos\theta$ is a trigonometric identity as it is true for all values of θ.

However, the equation $(\sin\theta + \cos\theta)^2 = 1$ is not an identity because it is true when $\theta = 0^\circ$, but not true when $\theta = 45^\circ$ as $(\sin 45^\circ + \cos 45^\circ)^2 = \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)^2 = 2 \neq 1$.

> In this chapter, all the trigonometric identities and equations are assumed to be well defined for those values of the variables for which they are meaningful.

Let us establish three useful identities called the Pythagorean identities and use them to obtain some other identities.

In the right-angled $\triangle ABC$, we have

$$AB^2 + BC^2 = AC^2 \qquad (1)$$

Dividing each term of (1) by AC^2, we get

$$\frac{AB^2}{AC^2} + \frac{BC^2}{AC^2} = \frac{AC^2}{AC^2} \qquad (AC \neq 0)$$

$$\left(\frac{AB}{AC}\right)^2 + \left(\frac{BC}{AC}\right)^2 = 1$$

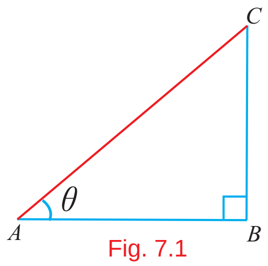


Fig. 7.1

Thus, $\cos^2 A + \sin^2 A = 1$

Let $\angle A = \theta$. Then for all $0^\circ < \theta < 90^\circ$ we have,

$$\cos^2\theta + \sin^2\theta = 1. \qquad (2)$$

Evidently, $\cos^2 0^\circ + \sin^2 0^\circ = 1$ and $\cos^2 90^\circ + \sin^2 90^\circ = 1$ and so (2) is true for all θ such that $0^\circ \leq \theta \leq 90^\circ$

Let us divide (1) by AB^2, we get

$$\frac{AB^2}{AB^2} + \frac{BC^2}{AB^2} = \left(\frac{AC}{AB}\right)^2 \qquad (\because AB \neq 0)$$

$$\left(\frac{AB}{AB}\right)^2 + \left(\frac{BC}{AB}\right)^2 = \left(\frac{AC}{AB}\right)^2 \implies 1 + \tan^2\theta = \sec^2\theta. \qquad (3)$$

Since $\tan\theta$ and $\sec\theta$ are not defined for $\theta = 90^\circ$, the identity (3) is true for all θ such that $0^\circ \leq \theta < 90^\circ$

Again dividing each term of (1) by BC^2, we get

$$\frac{AB^2}{BC^2} + \frac{BC^2}{BC^2} = \left(\frac{AC}{BC}\right)^2 \qquad (\because BC \neq 0)$$

$$\left(\frac{AB}{BC}\right)^2 + \left(\frac{BC}{BC}\right)^2 = \left(\frac{AC}{BC}\right)^2 \implies \cot^2\theta + 1 = \operatorname{cosec}^2\theta. \qquad (4)$$

Since $\cot\theta$ and $\operatorname{cosec}\theta$ are not defined for $\theta = 0^\circ$, the identity (4) is true for all θ such that $0^\circ < \theta \leq 90^\circ$

Some equal forms of identities from (2) to (4) are listed below.

	Identity	Equal forms
(i)	$\sin^2\theta + \cos^2\theta = 1$	$\sin^2\theta = 1 - \cos^2\theta$ (or) $\cos^2\theta = 1 - \sin^2\theta$
(ii)	$1 + \tan^2\theta = \sec^2\theta$	$\sec^2\theta - \tan^2\theta = 1$ (or) $\tan^2\theta = \sec^2\theta - 1$
(iii)	$1 + \cot^2\theta = \csc^2\theta$	$\csc^2\theta - \cot^2\theta = 1$ (or) $\cot^2\theta = \csc^2\theta - 1$

Remarks

We have proved the above identities for an acute angle θ. But these identities are true for any angle θ for which the trigonometric functions are meaningful. In this book we shall restrict ourselves to acute angles only.

In general, there is no common method for proving trigonometric identities involving trigonometric functions. However, some of the techniques listed below may be useful in proving trigonometric identities.

(i) Study the identity carefully, keeping in mind what is given and what you need to arrive.

(ii) Generally, the more complicated side of the identity may be taken first and simplified as it is easier to simplify than to expand or enlarge the simpler one.

(iii) If both sides of the identity are complicated, each may be taken individually and simplified independently of each other to the same expression.

(iv) Combine fractions using algebraic techniques for adding expressions.

(v) If necessary, change each term into their sine and cosine equivalents and then try to simplify.

(vi) If an identity contains terms involving $\tan^2\theta, \cot^2\theta, \csc^2\theta, \sec^2\theta$, it may be more helpful to use the results $\sec^2\theta = 1 + \tan^2\theta$ and $\csc^2\theta = 1 + \cot^2\theta$.

Example 7.1

Prove the identity $\dfrac{\sin\theta}{\csc\theta} + \dfrac{\cos\theta}{\sec\theta} = 1$

Solution

Now, $\dfrac{\sin\theta}{\csc\theta} + \dfrac{\cos\theta}{\sec\theta} = \dfrac{\sin\theta}{\left(\dfrac{1}{\sin\theta}\right)} + \dfrac{\cos\theta}{\left(\dfrac{1}{\cos\theta}\right)}$

$= \sin^2\theta + \cos^2\theta = 1$.

Example 7.2

Prove the identity $\sqrt{\dfrac{1-\cos\theta}{1+\cos\theta}} = \text{cosec}\,\theta - \cot\theta$

Solution

Consider $\sqrt{\dfrac{1-\cos\theta}{1+\cos\theta}} = \sqrt{\dfrac{(1-\cos\theta)}{(1+\cos\theta)} \times \dfrac{(1-\cos\theta)}{(1-\cos\theta)}}$

$$= \sqrt{\frac{(1-\cos\theta)^2}{1^2-\cos^2\theta}} = \sqrt{\frac{(1-\cos\theta)^2}{\sin^2\theta}} \qquad (1-\cos^2\theta = \sin^2\theta)$$

$$= \frac{1-\cos\theta}{\sin\theta} = \frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta}$$

$$= \text{cosec}\,\theta - \cot\theta\,.$$

Example 7.3

Prove the identity $[\text{cosec}(90^\circ-\theta)-\sin(90^\circ-\theta)][\text{cosec}\,\theta-\sin\theta][\tan\theta+\cot\theta]=1$

Solution

Now, $[\text{cosec}(90^\circ-\theta)-\sin(90^\circ-\theta)][\text{cosec}\,\theta-\sin\theta][\tan\theta+\cot\theta]$

$$= (\sec\theta-\cos\theta)(\text{cosec}\,\theta-\sin\theta)\left(\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}\right) \qquad \because\ \text{cosec}(90^\circ-\theta)=\sec\theta,\quad \because\ \sin(90^\circ-\theta)=\cos\theta$$

$$= \left(\frac{1}{\cos\theta}-\cos\theta\right)\left(\frac{1}{\sin\theta}-\sin\theta\right)\left(\frac{\sin^2\theta+\cos^2\theta}{\sin\theta\cos\theta}\right)$$

$$= \left(\frac{1-\cos^2\theta}{\cos\theta}\right)\left(\frac{1-\sin^2\theta}{\sin\theta}\right)\left(\frac{1}{\sin\theta\cos\theta}\right)$$

$$= \left(\frac{\sin^2\theta}{\cos\theta}\right)\left(\frac{\cos^2\theta}{\sin\theta}\right)\left(\frac{1}{\sin\theta\cos\theta}\right) = 1$$

Example 7.4

Prove that $\dfrac{\tan\theta+\sec\theta-1}{\tan\theta-\sec\theta+1} = \dfrac{1+\sin\theta}{\cos\theta}$

Solution

We consider $\dfrac{\tan\theta+\sec\theta-1}{\tan\theta-\sec\theta+1}$

$$= \frac{\tan\theta+\sec\theta-(\sec^2\theta-\tan^2\theta)}{\tan\theta-\sec\theta+1} \qquad (\sec^2\theta-\tan^2\theta=1)$$

$$= \frac{(\tan\theta+\sec\theta)-(\sec\theta+\tan\theta)(\sec\theta-\tan\theta)}{\tan\theta-\sec\theta+1} \qquad (a^2-b^2=(a+b)(a-b))$$

$$= \frac{(\tan\theta+\sec\theta)[1-(\sec\theta-\tan\theta)]}{\tan\theta-\sec\theta+1}$$

$$= \frac{(\tan\theta+\sec\theta)(\tan\theta-\sec\theta+1)}{\tan\theta-\sec\theta+1}$$

$$= \tan\theta+\sec\theta = \frac{\sin\theta}{\cos\theta}+\frac{1}{\cos\theta} = \frac{1+\sin\theta}{\cos\theta}$$

Example 7.5

Prove the identity $\dfrac{\tan\theta}{1-\cot\theta}+\dfrac{\cot\theta}{1-\tan\theta}=1+\tan\theta+\cot\theta.$

Solution

Now, $\dfrac{\tan\theta}{1-\cot\theta}+\dfrac{\cot\theta}{1-\tan\theta}$

$$=\frac{\tan\theta}{1-\dfrac{1}{\tan\theta}}+\frac{\dfrac{1}{\tan\theta}}{1-\tan\theta}=\frac{\tan\theta}{\dfrac{\tan\theta-1}{\tan\theta}}+\frac{\dfrac{1}{\tan\theta}}{1-\tan\theta}$$

$$=\frac{\tan^2\theta}{\tan\theta-1}+\frac{1}{\tan\theta(1-\tan\theta)}=\frac{\tan^2\theta}{\tan\theta-1}+\frac{1}{(-\tan\theta)(\tan\theta-1)}$$

$$=\frac{\tan^2\theta}{\tan\theta-1}-\frac{1}{(\tan\theta)(\tan\theta-1)}$$

$$=\frac{1}{(\tan\theta-1)}\left(\tan^2\theta-\frac{1}{\tan\theta}\right)$$

$$=\frac{1}{(\tan\theta-1)}\quad\frac{(\tan^3\theta-1)}{\tan\theta}$$

$$=\frac{(\tan\theta-1)(\tan^2\theta+\tan\theta+1^2)}{(\tan\theta-1)\tan\theta}\qquad(\because a^3-b^3=(a-b)(a^2+ab+b^2))$$

$$=\frac{\tan^2\theta+\tan\theta+1}{\tan\theta}$$

$$=\frac{\tan^2\theta}{\tan\theta}+\frac{\tan\theta}{\tan\theta}+\frac{1}{\tan\theta}=\tan\theta+1+\cot\theta$$

$$=1+\tan\theta+\cot\theta.$$

Example 7.6

Prove the identity

$$(\sin\theta+\operatorname{cosec}\theta)^2+(\cos\theta+\sec\theta)^2=7+\tan^2\theta+\cot^2\theta.$$

Solution

Let us consider $(\sin\theta+\operatorname{cosec}\theta)^2+(\cos\theta+\sec\theta)^2$

$$=\sin^2\theta+\operatorname{cosec}^2\theta+2\sin\theta\operatorname{cosec}\theta+\cos^2\theta+\sec^2\theta+2\cos\theta\sec\theta$$

$$=\sin^2\theta+\cos^2\theta+\operatorname{cosec}^2\theta+\sec^2\theta+2\sin\theta\frac{1}{\sin\theta}+2\cos\theta\frac{1}{\cos\theta}$$

$$=1+(1+\cot^2\theta)+(1+\tan^2\theta)+2+2$$

$$=7+\tan^2\theta+\cot^2\theta.$$

Example 7.7

Prove the identity $(\sin^6\theta + \cos^6\theta) = 1 - 3\sin^2\theta\cos^2\theta$.

Solution

Now $\quad \sin^6\theta + \cos^6\theta$

$$= (\sin^2\theta)^3 + (\cos^2\theta)^3$$

$$= (\sin^2\theta + \cos^2\theta)^3 - 3\sin^2\theta\cos^2\theta(\sin^2\theta + \cos^2\theta)$$

$$(a^3 + b^3 = (a+b)^3 - 3ab(a+b))$$

$$= 1 - 3\sin^2\theta\cos^2\theta. \qquad (\sin^2\theta + \cos^2\theta = 1)$$

Example 7.8

Prove the identity $\dfrac{\sin\theta - 2\sin^3\theta}{2\cos^3\theta - \cos\theta} = \tan\theta$.

Solution

Now, $\dfrac{\sin\theta - 2\sin^3\theta}{2\cos^3\theta - \cos\theta} = \dfrac{\sin\theta(1 - 2\sin^2\theta)}{\cos\theta(2\cos^2\theta - 1)}$

$$= \left(\frac{\sin\theta}{\cos\theta}\right)\left(\frac{\sin^2\theta + \cos^2\theta - 2\sin^2\theta}{2\cos^2\theta - (\sin^2\theta + \cos^2\theta)}\right) \qquad (\sin^2\theta + \cos^2\theta = 1)$$

$$= (\tan\theta)\left(\frac{\cos^2\theta - \sin^2\theta}{\cos^2\theta - \sin^2\theta}\right) = \tan\theta.$$

Example 7.9

Prove the identity $\dfrac{\sec\theta - \tan\theta}{\sec\theta + \tan\theta} = 1 - 2\sec\theta\tan\theta + 2\tan^2\theta$.

Solution

We consider $\dfrac{\sec\theta - \tan\theta}{\sec\theta + \tan\theta}$

$$= \left(\frac{\sec\theta - \tan\theta}{\sec\theta + \tan\theta}\right)\times\left(\frac{\sec\theta - \tan\theta}{\sec\theta - \tan\theta}\right)$$

$$= \frac{(\sec\theta - \tan\theta)^2}{\sec^2\theta - \tan^2\theta}$$

$$= \frac{(\sec\theta - \tan\theta)^2}{1} \qquad (\sec^2\theta - \tan^2\theta = 1)$$

$$= (\sec\theta - \tan\theta)^2 = \sec^2\theta + \tan^2\theta - 2\sec\theta\tan\theta$$

$$= (1 + \tan^2\theta) + \tan^2\theta - 2\sec\theta\tan\theta \qquad (\sec^2\theta = 1 + \tan^2\theta)$$

$$= 1 - 2\sec\theta\tan\theta + 2\tan^2\theta.$$

Example 7.10

Prove that $\dfrac{1+\sec\theta}{\sec\theta} = \dfrac{\sin^2\theta}{1-\cos\theta}$.

Solution

First, we consider $\dfrac{1+\sec\theta}{\sec\theta}$

$$= \frac{1+\frac{1}{\cos\theta}}{\frac{1}{\cos\theta}} = \frac{(\cos\theta+1)}{\cos\theta}(\cos\theta)$$

$$= 1+\cos\theta$$

$$= (1+\cos\theta)\times\frac{(1-\cos\theta)}{(1-\cos\theta)}$$

$$= \frac{1-\cos^2\theta}{1-\cos\theta}$$

$$= \frac{\sin^2\theta}{1-\cos\theta}.$$

Example 7.11

Prove the identity $(\operatorname{cosec}\theta-\sin\theta)(\sec\theta-\cos\theta) = \dfrac{1}{\tan\theta+\cot\theta}$.

Solution

Now, $(\operatorname{cosec}\theta-\sin\theta)(\sec\theta-\cos\theta)$

$$= \left(\frac{1}{\sin\theta}-\sin\theta\right)\left(\frac{1}{\cos\theta}-\cos\theta\right)$$

$$= \left(\frac{1-\sin^2\theta}{\sin\theta}\right)\left(\frac{1-\cos^2\theta}{\cos\theta}\right)$$

$$= \frac{\cos^2\theta}{\sin\theta}\frac{\sin^2\theta}{\cos\theta} = \sin\theta\cos\theta \qquad (1)$$

Next, consider $\dfrac{1}{\tan\theta+\cot\theta}$

$$= \frac{1}{\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}}$$

$$= \frac{1}{\left(\frac{\sin^2\theta+\cos^2\theta}{\sin\theta\cos\theta}\right)}$$

$$= \sin\theta\cos\theta \qquad (2)$$

Note

$$\sin\theta\cos\theta = \frac{\sin\theta\cos\theta}{1}$$

$$= \frac{\sin\theta\cos\theta}{\sin^2\theta+\cos^2\theta}$$

$$= \frac{1}{\frac{\sin^2\theta+\cos^2\theta}{\sin\theta\cos\theta}}$$

$$= \frac{1}{\frac{\sin^2\theta}{\sin\theta\cos\theta}+\frac{\cos^2\theta}{\sin\theta\cos\theta}}$$

$$= \frac{1}{\tan\theta+\cot\theta}$$

From (1) and (2), we get

$$(\operatorname{cosec}\theta-\sin\theta)(\sec\theta-\cos\theta) = \frac{1}{\tan\theta+\cot\theta}.$$

Example 7.12

If $\tan\theta + \sin\theta = m$, $\tan\theta - \sin\theta = n$ and $m \neq n$, then show that $m^2 - n^2 = 4\sqrt{mn}$.

Solution

Given that $\quad m = \tan\theta + \sin\theta$ and $n = \tan\theta - \sin\theta$.

Now, $\quad m^2 - n^2 = (\tan\theta + \sin\theta)^2 - (\tan\theta - \sin\theta)^2$

$$= \tan^2\theta + \sin^2\theta + 2\sin\theta\tan\theta - (\tan^2\theta + \sin^2\theta - 2\sin\theta\tan\theta)$$

$$= 4\sin\theta\tan\theta \qquad (1)$$

Also, $\quad 4\sqrt{mn} = 4\sqrt{(\tan\theta + \sin\theta)(\tan\theta - \sin\theta)}$

$$= 4\sqrt{\tan^2\theta - \sin^2\theta} = 4\sqrt{\left(\frac{\sin^2\theta}{\cos^2\theta} - \sin^2\theta\right)}$$

$$= 4\sqrt{\sin^2\theta\left(\frac{1}{\cos^2\theta} - 1\right)}$$

$$= 4\sqrt{\sin^2\theta(\sec^2\theta - 1)} = 4\sqrt{\sin^2\theta\tan^2\theta} \quad (\because \sec^2\theta - 1 = \tan^2\theta)$$

$$= 4\sin\theta\tan\theta. \qquad (2)$$

From (1) and (2), we get $m^2 - n^2 = 4\sqrt{mn}$.

Example 7.13

If $\tan^2\alpha = \cos^2\beta - \sin^2\beta$, then prove that $\cos^2\alpha - \sin^2\alpha = \tan^2\beta$.

Solution

Given that $\quad \cos^2\beta - \sin^2\beta = \tan^2\alpha$

$$\frac{\cos^2\beta - \sin^2\beta}{1} = \frac{\sin^2\alpha}{\cos^2\alpha}$$

$$\frac{\cos^2\beta - \sin^2\beta}{\cos^2\beta + \sin^2\beta} = \frac{\sin^2\alpha}{\cos^2\alpha}$$

> Componendo and dividendo rule
>
> If $\dfrac{a}{b} = \dfrac{c}{d}$, then $\dfrac{a+b}{a-b} = \dfrac{c+d}{c-d}$

Applying componendo and dividendo rule, we get

$$\frac{(\cos^2\beta - \sin^2\beta) + (\cos^2\beta + \sin^2\beta)}{(\cos^2\beta - \sin^2\beta) - (\cos^2\beta + \sin^2\beta)} = \frac{\sin^2\alpha + \cos^2\alpha}{\sin^2\alpha - \cos^2\alpha}$$

$$\Longrightarrow \quad \frac{2\cos^2\beta}{-2\sin^2\beta} = \frac{1}{\sin^2\alpha - \cos^2\alpha}$$

$$\Longrightarrow \quad -\frac{\sin^2\beta}{\cos^2\beta} = \sin^2\alpha - \cos^2\alpha$$

$\Longrightarrow \quad \tan^2\beta = \cos^2\alpha - \sin^2\alpha$, which completes the proof.

Note: This problem can also be solved without using componendo and dividendo rule.

Exercise 7.1

1. Determine whether each of the following is an identity or not.

(i) $\cos^2\theta + \sec^2\theta = 2 + \sin\theta$ (ii) $\cot^2\theta + \cos\theta = \sin^2\theta$

2. Prove the following identities

(i) $\sec^2\theta + \operatorname{cosec}^2\theta = \sec^2\theta \operatorname{cosec}^2\theta$ (ii) $\dfrac{\sin\theta}{1-\cos\theta} = \operatorname{cosec}\theta + \cot\theta$

(iii) $\sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}} = \sec\theta - \tan\theta$ (iv) $\dfrac{\cos\theta}{\sec\theta - \tan\theta} = 1 + \sin\theta$

(v) $\sqrt{\sec^2\theta + \operatorname{cosec}^2\theta} = \tan\theta + \cot\theta$ (vi) $\dfrac{1+\cos\theta - \sin^2\theta}{\sin\theta(1+\cos\theta)} = \cot\theta$

(vii) $\sec\theta(1-\sin\theta)(\sec\theta + \tan\theta) = 1$ (viii) $\dfrac{\sin\theta}{\operatorname{cosec}\theta + \cot\theta} = 1 - \cos\theta$

3. Prove the following identities.

(i) $\dfrac{\sin(90^\circ - \theta)}{1+\sin\theta} + \dfrac{\cos\theta}{1-\cos(90^\circ - \theta)} = 2\sec\theta$

(ii) $\dfrac{\tan\theta}{1-\cot\theta} + \dfrac{\cot\theta}{1-\tan\theta} = 1 + \sec\theta\operatorname{cosec}\theta$

(iii) $\dfrac{\sin(90^0 - \theta)}{1-\tan\theta} + \dfrac{\cos(90^0-\theta)}{1-\cot\theta} = \cos\theta + \sin\theta$

(iv) $\dfrac{\tan(90^0-\theta)}{\operatorname{cosec}\theta + 1} + \dfrac{\operatorname{cosec}\theta + 1}{\cot\theta} = 2\sec\theta.$

(v) $\dfrac{\cot\theta + \operatorname{cosec}\theta - 1}{\cot\theta - \operatorname{cosec}\theta + 1} = \operatorname{cosec}\theta + \cot\theta.$

(vi) $(1+\cot\theta - \operatorname{cosec}\theta)(1+\tan\theta + \sec\theta) = 2$

(vii) $\dfrac{\sin\theta - \cos\theta + 1}{\sin\theta + \cos\theta - 1} = \dfrac{1}{\sec\theta - \tan\theta}$

(viii) $\dfrac{\tan\theta}{1-\tan^2\theta} = \dfrac{\sin\theta\sin(90^0-\theta)}{2\sin^2(90^0-\theta) - 1}$

(ix) $\dfrac{1}{\operatorname{cosec}\theta - \cot\theta} - \dfrac{1}{\sin\theta} = \dfrac{1}{\sin\theta} - \dfrac{1}{\operatorname{cosec}\theta + \cot\theta}.$

(x) $\dfrac{\cot^2\theta + \sec^2\theta}{\tan^2\theta + \operatorname{cosec}^2\theta} = (\sin\theta\cos\theta)(\tan\theta + \cot\theta).$

4. If $x = a\sec\theta + b\tan\theta$ and $y = a\tan\theta + b\sec\theta$, then prove that $x^2 - y^2 = a^2 - b^2$.

5. If $\tan\theta = n\tan\alpha$ and $\sin\theta = m\sin\alpha$, then prove that $\cos^2\theta = \dfrac{m^2-1}{n^2-1}$, $n \neq \pm 1$.

6. If $\sin\theta$, $\cos\theta$ and $\tan\theta$ are in G.P., then prove that $\cot^6\theta - \cot^2\theta = 1$.

7.3 Heights and Distances

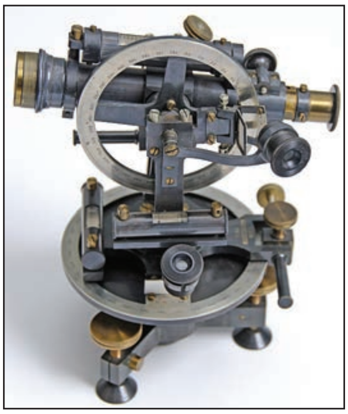


Fig. 7.2

One wonders, how the distance between planets, height of Mount Everest, distance between two objects which are far off like Earth and Sun ..., are measured or calculated. Can these be done with measuring tapes?

Of course, it is impossible to do so. Quite interestingly such distances are calculated using the idea of trigonometric ratios. These ratios are also used to construct maps, determine the position of an Island in relation to longitude and latitude.

A theodolite (Fig. 7.2) is an instrument which is used in measuring the angle between an object and the eye of the observer. A theodolite consists of two graduated wheels placed at right angles to each other and a telescope. The wheels are used for the measurement of horizontal and vertical angles. The angle to the desired point is measured by positioning the telescope towards that point. The angle can be read on the telescopic scale.

Suppose we wish to find the height of our school flag post without actually measuring it.

Assume that a student stands on the ground at point *A*, which is 10 m away from the foot *B* of the flag post. He observes the top of the flag post at an angle of 60°. Suppose that the height of his eye level *E* from the ground level is $1.2\,\text{m}$. (see fig no.7.3)

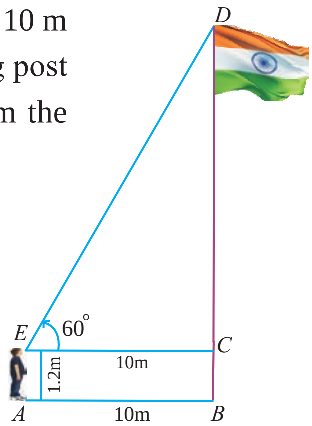


Fig. 7.3

In the right angled ΔDEC, $\angle DEC = 60^\circ$.

Now, $\tan 60^\circ = \dfrac{CD}{EC}$

$\Longrightarrow$ $CD = EC \tan 60^\circ$

Thus, $CD = 10\sqrt{3} = 10\times 1.732$

$= 17.32\,\text{m}$

Hence, the height of the flag post, $BD = BC + CD$

$= 1.2 + 17.32 = 18.52\,\text{m}$

Thus, we are able to find the height of our school flag post without actually measuring it. So, in a right triangle, if one side and one acute angle are known, we can find the other sides of the triangle using trigonometrical ratios. Let us define a few terms which we use very often in finding the heights and distances.

Line of sight

If we are viewing an object, the line of sight is a straight line from our eye to the object. Here we treat the object as a point since distance involved is quite large.

Angle of depression and angle of elevation

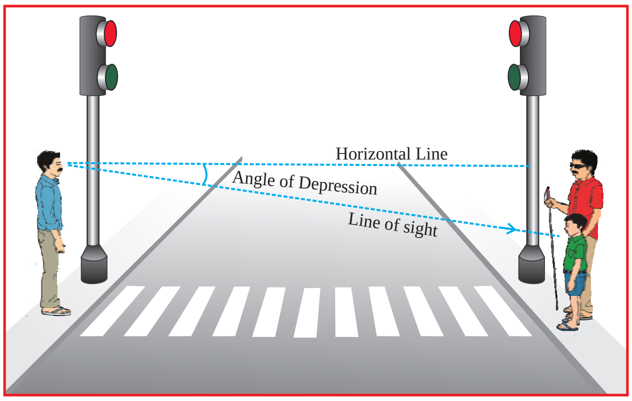


Fig. 7.4

If an object is below the horizontal line from the eye, we have to lower our head to view the object. In this process our eyes moves through an angle. This angle is called the angle of depression, That is, the angle of depression of an object viewed is the angle formed by the line of sight with the horizontal line, when the object is below the horizontal line (See Fig. 7.4).

If an object is above the horizontal line from our eyes we have to raise our head to view the object. In this process our eyes move through an angle formed by the line of sight and horizontal line which is called the angle of elevation. (See Fig. 7.5).

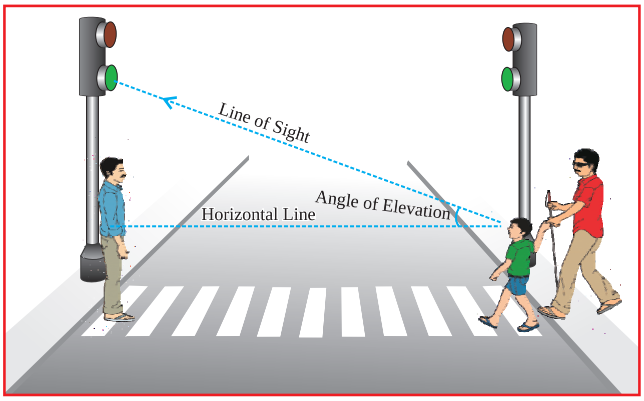


Fig. 7.5

Note

(i) An observer is taken as a point if the height of the observer is not given.

(ii) The angle of elevation of an object as seen by the observer is same as the angle of depression of the observer as seen from the object.

To solve problems involving heights and distances, the following strategy may be useful

(i) Read the statements of the question carefully and draw a rough diagram accordingly.

(ii) Label the diagram and mark the given values.

(iii) Denote the unknown dimension, say 'h' when the height is to be calculated and 'x' when the distance is to be calculated.

(iv) Identify the trigonometrical ratio that will be useful for solving the problem.

(v) Substitute the given values and solve for unknown.

The following activity may help us learn how to measure the height of an object which will be difficult to measure otherwise.

Activity

- Tie one end of a string to the middle of a straw and the other end of the string to a paper clip.
- Glue this straw to the base of a protractor so that the middle of the straw aligns with the centre of the protractor. Make sure that the string hangs freely to create a vertical line or the plumb-line.
- Find an object outside that is too tall to measure directly, such as a basket ball hoop, a flagpole, or the school building.

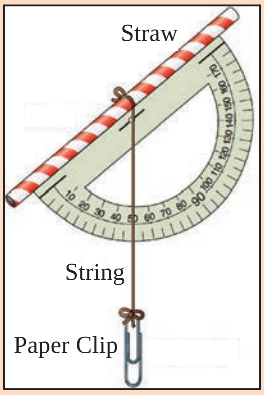


Fig. 7.6

- Look at the top of the object through the straw. Find the angle where the string and protractor intersect. Determine the angle of elevation by subtracting this measurement from 90°. Let it be θ.
- Measure the distance from your eye level to the ground and from your foot to the base of the object that you are measuring, say y.
- Make a sketch of your measurements.
- To find the height (h) of the object, use the following equation.

  $h = x + y\tan\theta$, where x represents the distance from your eye level to the ground.

Example 7.14

A kite is flying with a string of length 200 m. If the thread makes an angle 30° with the ground, find the distance of the kite from the ground level. (Here, assume that the string is along a straight line)

Solution Let h denote the distance of the kite from the ground level.

In the figure, AC is the string

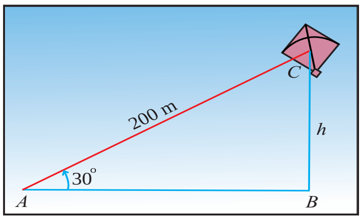


Fig. 7.7

Given that $\angle CAB = 30^\circ$ and $AC = 200$ m.

In the right $\triangle\, CAB$, $\sin 30^\circ = \dfrac{h}{200}$

$$\Longrightarrow \quad h = 200\sin 30^\circ$$

$$\therefore \quad h = 200 \times \frac{1}{2} = 100 \text{ m}$$

Hence, the distance of the kite from the ground level is 100 m.

Example 7.15

A ladder leaning against a vertical wall, makes an angle of 60° with the ground. The foot of the ladder is 3.5m away from the wall. Find the length of the ladder.

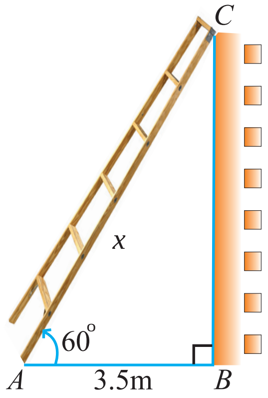


Fig. 7.8

Solution Let AC denote the ladder and B be the foot of the wall.

Let the length of the ladder AC be x metres.

Given that $\angle CAB = 60^\circ$ and $AB = 3.5$ m.

In the right $\triangle CAB$, $\cos 60^\circ = \dfrac{AB}{AC}$

$$\Longrightarrow \quad AC = \frac{AB}{\cos 60^\circ}$$

$$\therefore \quad x = 2 \times 3.5 = 7 \text{ m}$$

Thus, the length of the ladder is 7m.

Example 7.16

Find the angular elevation (angle of elevation from the ground level) of the Sun when the length of the shadow of a 30 m long pole is $10\sqrt{3}$ m.

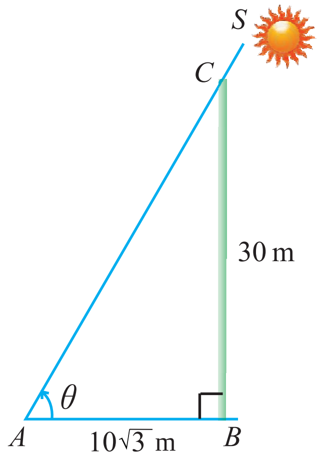


Fig. 7.9

Solution Let S be the position of the Sun and BC be the pole.

Let AB denote the length of the shadow of the pole.

Let the angular elevation of the Sun be θ.

Given that $AB = 10\sqrt{3}$ m and

$BC = 30$ m

In the right $\triangle CAB$, $\tan\theta = \dfrac{BC}{AB} = \dfrac{30}{10\sqrt{3}} = \dfrac{3}{\sqrt{3}}$

$$\Longrightarrow \quad \tan\theta = \sqrt{3}$$

$$\therefore \quad \theta = 60^\circ$$

Thus, the angular elevation of the Sun from the ground level is 60°.

Example 7.17

The angle of elevation of the top of a tower as seen by an observer is 30°. The observer is at a distance of $30\sqrt{3}$ m from the tower. If the eye level of the observer is 1.5 m above the ground level, then find the height of the tower.

Solution Let BD be the height of the tower and AE be the distance of the eye level of the observer from the ground level.

Draw EC parallel to AB such that $AB = EC$.

Given $AB = EC = 30\sqrt{3}$ m and

$AE = BC = 1.5$m

In right angled $\triangle DEC$,

$$\tan 30^\circ = \frac{CD}{EC}$$

$$\Longrightarrow \quad CD = EC \tan 30^\circ = \frac{30\sqrt{3}}{\sqrt{3}}$$

$$\therefore \quad CD = 30\,\text{m}$$

Thus, the height of the tower, $BD = BC + CD$

$$= 1.5 + 30 = 31.5\,\text{m}.$$

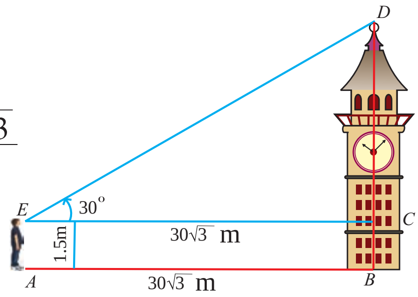


Fig. 7.10

Example 7.18

A vertical tree is broken by the wind. The top of the tree touches the ground and makes an angle 30° with it. If the top of the tree touches the ground 30 m away from its foot, then find the actual height of the tree.

Solution Let C be the point at which the tree is broken and let the top of the tree touch the ground at A.

Let B denote the foot of the tree.

Given $AB = 30$ m and

$\angle CAB = 30^\circ$.

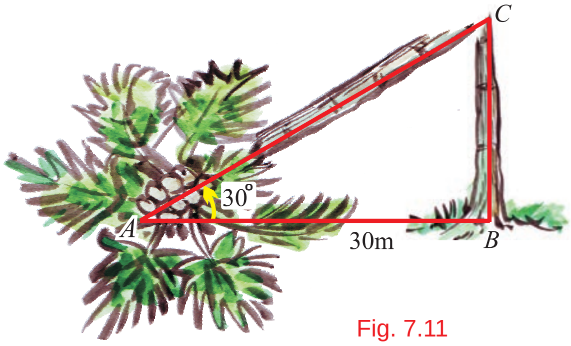


Fig. 7.11

In the right angled $\triangle CAB$,

$$\tan 30^\circ = \frac{BC}{AB}$$

$$\Longrightarrow \quad BC = AB \tan 30^\circ$$

$$\therefore \quad BC = \frac{30}{\sqrt{3}}$$

$$= 10\sqrt{3}\,\text{m} \qquad (1)$$

Now, $$\cos 30^\circ = \frac{AB}{AC}$$

$$\Longrightarrow \quad AC = \frac{AB}{\cos 30^\circ}$$

So, $$AC = \frac{30 \times 2}{\sqrt{3}} = 10\sqrt{3} \times 2 = 20\sqrt{3}\,\text{m}. \qquad (2)$$

Thus, the height of the tree $= BC + AC = 10\sqrt{3} + 20\sqrt{3}$

$$= 30\sqrt{3}\,\text{m}.$$

Example 7.19

A jet fighter at a height of 3000 m from the ground, passes directly over another jet fighter at an instance when their angles of elevation from the same observation point are 60° and 45° respectively. Find the distance of the first jet fighter from the second jet at that instant. $(\sqrt{3} = 1.732)$

Solution Let O be the point of observation.

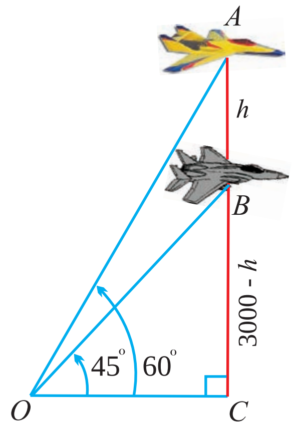


Fig. 7.12

Let A and B be the positions of the two jet fighters at the given instant when one is directly above the other.

Let C be the point on the ground such that $AC = 3000$ m.

Given $\angle AOC = 60^\circ$ and $\angle BOC = 45^\circ$

Let h denote the distance between the jets at the instant.

In the right angled $\triangle BOC$, $\tan 45^\circ = \dfrac{BC}{OC}$

$$\Longrightarrow \quad OC = BC \qquad (\because \tan 45^\circ = 1)$$

Thus, $\quad OC = 3000 - h \qquad (1)$

In the right angled $\triangle AOC$, $\tan 60^\circ = \dfrac{AC}{OC}$

$$\Longrightarrow \quad OC = \frac{AC}{\tan 60^\circ} = \frac{3000}{\sqrt{3}}$$

$$= \frac{3000}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = 1000\sqrt{3} \qquad (2)$$

From (1) and (2), we get $\quad 3000 - h = 1000\sqrt{3}$

$$\Longrightarrow \quad h = 3000 - 1000 \times 1.732 = 1268\,\text{m}$$

The distance of the first jet fighter from the second jet at that instant is 1268 m.

Example 7.20

The angle of elevation of the top of a hill from the foot of a tower is 60° and the angle of elevation of the top of the tower from the foot of the hill is 30°. If the tower is 50 m high, then find the height of the hill.

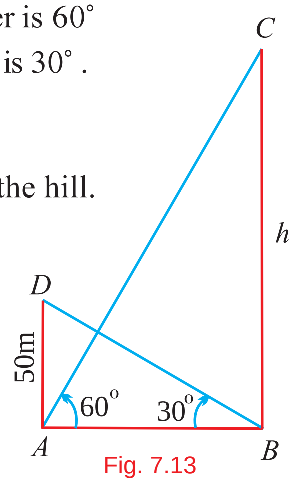


Fig. 7.13

Solution Let AD be the height of tower and BC be the height of the hill.

Given $\angle CAB = 60^\circ$, $\angle ABD = 30^\circ$ and $AD = 50$m.

Let $BC = h$ metres.

Now, in the right angled ΔDAB, $\tan 30^\circ = \dfrac{AD}{AB}$

$$\Longrightarrow \quad AB = \frac{AD}{\tan 30^\circ}$$

$$\therefore \quad AB = 50\sqrt{3}\,\text{m} \qquad (1)$$

Also, in the right angled $\triangle CAB$, $\tan 60^\circ = \dfrac{BC}{AB}$

$$\Longrightarrow \quad BC = AB \tan 60^\circ$$

Thus, using (1) we get $\quad h = BC = (50\sqrt{3})\sqrt{3} = 150\,\text{m}$

Hence, the height of the hill is 150 m.

Example 7.21

A vertical wall and a tower are on the ground. As seen from the top of the tower, the angles of depression of the top and bottom of the wall are 45° and 60° respectively. Find the height of the wall if the height of the tower is 90 m. ($\sqrt{3} = 1.732$)

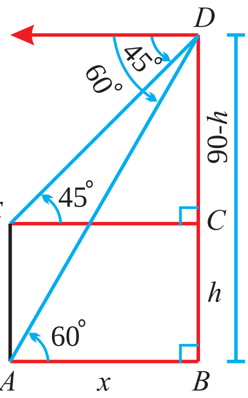


Fig. 7.14

Solution Let AE denote the wall and BD denote the tower.

Draw EC parallel to AB such that $AB=EC$. Thus, $AE = BC$.

Let $AB = x$ metres and $AE = h$ metres.

Given that $BD = 90$ m and $\angle DAB = 60^\circ$, $\angle DEC = 45^\circ$.

Now, $AE = BC = h$ metres

Thus, $CD = BD - BC = 90 - h$.

In the right angled $\triangle DAB$, $\tan 60^\circ = \dfrac{BD}{AB} = \dfrac{90}{x}$

$$\Longrightarrow \quad x = \frac{90}{\sqrt{3}} = 30\sqrt{3} \qquad (1)$$

In the right angled $\triangle DEC$, $\tan 45^\circ = \dfrac{DC}{EC} = \dfrac{90-h}{x}$

$$\text{Thus,} \quad x = 90 - h \qquad (2)$$

From (1) and (2), we have $90 - h = 30\sqrt{3}$

Thus, the height of the wall, $h = 90 - 30\sqrt{3} = 38.04$ m.

Example 7.22

A girl standing on a lighthouse built on a cliff near the seashore, observes two boats due East of the lighthouse. The angles of depression of the two boats are 30° and 60°. The distance between the boats is 300 m. Find the distance of the top of the lighthouse from the sea level. (Boats and foot of the lighthouse are in a straight line)

Solution Let A and D denote the foot of the cliff and the top of the lighthouse respectively. Let B and C denote the two boats.

Let h metres be the distance of the top of the lighthouse from the sea level.

Let $AB = x$ metres.

Given that $\angle ABD = 60^\circ, \angle ACD = 30^\circ$

In the right angled ΔABD,

$$\tan 60^\circ = \frac{AD}{AB}$$

$$\Longrightarrow \quad AB = \frac{AD}{\tan 60^\circ}$$

Thus, $$x = \frac{h}{\sqrt{3}} \qquad (1)$$

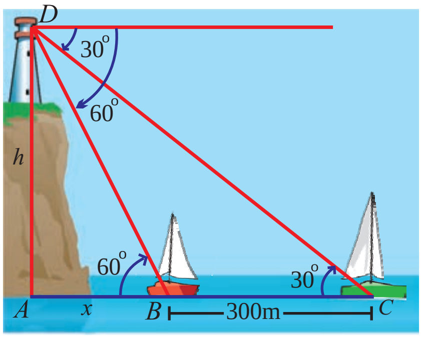


Fig. 7.15

Also, in the right angled $\triangle ACD$, we have

$$\tan 30^\circ = \frac{AD}{AC}$$

$$\Longrightarrow \quad AC = \frac{AD}{\tan 30^\circ} \Longrightarrow x + 300 = \frac{h}{\left(\frac{1}{\sqrt{3}}\right)}$$

Thus, $\quad x + 300 = h\sqrt{3}$. (2)

Using (1) in (2), we get $\frac{h}{\sqrt{3}} + 300 = h\sqrt{3}$

$$\Longrightarrow \quad h\sqrt{3} - \frac{h}{\sqrt{3}} = 300$$

$\therefore \quad 2h = 300\sqrt{3}$. Thus, $h = 150\sqrt{3}$.

Hence, the height of the lighthouse from the sea level is $150\sqrt{3}$ m.

Example 7.23

A boy spots a balloon moving with the wind in a horizontal line at a height of 88.2 m from the ground level. The distance of his eye level from the ground is 1.2 m. The angle of elevation of the balloon from his eyes at an instant is 60°. After some time, from the same point of observation, the angle of elevation of the balloon reduces to 30°. Find the distance covered by the balloon during the interval.

Solution Let A be the point of observation.

Let E and D be the positions of the balloon when its angles of elevation are 60° and 30° respectively.

Let B and C be the points on the horizontal line such that $BE = CD$.

Let A', B' and C' be the points on the ground such that

$$A'A = B'B = C'C = 1.2 \text{ m}.$$

Given that $\angle EAB = 60^\circ$, $\angle DAC = 30^\circ$

$$BB' = CC' = 1.2 \text{ m and } C'D = 88.2 \text{ m}.$$

Also, we have $\quad BE = CD = 87$ m.

Now, in the right angled $\triangle EAB$, we have

$$\tan 60^\circ = \frac{BE}{AB}$$

Thus, $\quad AB = \frac{87}{\tan 60^\circ} = \frac{87}{\sqrt{3}} = 29\sqrt{3}$

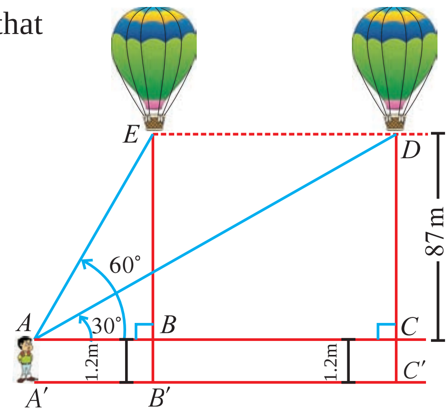


Fig. 7.16

Again in the right angled $\triangle DAC$, we have $\tan 30^\circ = \frac{DC}{AC}$

Thus, $\quad AC = \frac{87}{\tan 30^\circ} = 87\sqrt{3}$.

Therefore, the distance covered by the balloon is

$$ED = BC = AC - AB$$

$$= 87\sqrt{3} - 29\sqrt{3} = 58\sqrt{3} \text{ m}.$$

Example 7.24

A flag post stands on the top of a building. From a point on the ground, the angles of elevation of the top and bottom of the flag post are 60° and 45° respectively. If the height of the flag post is $10\,\text{m}$, find the height of the building. ($\sqrt{3} = 1.732$)

Solution

Let A be the point of observation and B be the foot of the building.

Let BC denote the height of the building and CD denote height of the flag post.

Given that $\angle CAB = 45^\circ$, $\angle DAB = 60^\circ$ and $CD = 10\,\text{m}$

Let $BC = h$ metres and $AB = x$ metres.

Now, in the right angled $\triangle CAB$,

$$\tan 45^\circ = \frac{BC}{AB}.$$

Thus, $AB = BC$ i.e., $x = h$ (1)

Also, in the right angled $\triangle DAB$,

$$\tan 60^\circ = \frac{BD}{AB}$$

$$\Longrightarrow \quad AB = \frac{h+10}{\tan 60^\circ} \quad \Longrightarrow x = \frac{h+10}{\sqrt{3}} \qquad (2)$$

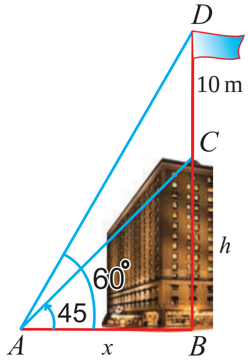


Fig. 7.17

From (1) and (2), we get

$$h = \frac{h+10}{\sqrt{3}}$$

$$\Longrightarrow \quad \sqrt{3}h - h = 10$$

$$\Longrightarrow \quad h = \left(\frac{10}{\sqrt{3}-1}\right)\left(\frac{\sqrt{3}+1}{\sqrt{3}+1}\right) = \frac{10(\sqrt{3}+1)}{3-1}$$

$$= 5\,(2.732) = 13.66\,\text{m}$$

Hence, the height of the building is $13.66\,\text{m}$.

Example 7.25

A man on the deck of a ship, $14\,\text{m}$ above the water level, observes that the angle of elevation of the top of a cliff is 60° and the angle of depression of the base of the cliff is 30°. Find the height of the cliff.

Solution Let BD be the height of the cliff.

Let A be the position of ship and E be the point of observation so that $AE = 14\,\text{m}$.

Draw EC parallel to AB such that $AB = EC$.

Given that $\angle ABE = 30^\circ, \angle DEC = 60^\circ$

In the right angled $\triangle ABE$, $\tan 30^\circ = \dfrac{AE}{AB}$

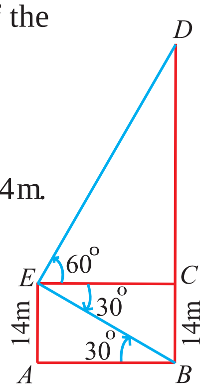


Fig. 7.18

$$\therefore \qquad AB = \frac{AE}{\tan 30^\circ} \implies AB = 14\sqrt{3}$$

Thus, $\quad EC = 14\sqrt{3} \qquad (\because\ AB = EC)$

In the right angled $\triangle DEC$, $\quad \tan 60^\circ = \frac{CD}{EC}$

$$\therefore\ CD = EC \tan 60^\circ \implies CD = (14\sqrt{3})\sqrt{3} = 42\text{ m}$$

Thus, the height of the cliff, $BD = BC + CD = 14 + 42 = 56$ m.

Example 7.26

The angle of elevation of an aeroplane from a point A on the ground is 60°. After a flight of 15 seconds horizontally, the angle of elevation changes to 30°. If the aeroplane is flying at a speed of 200 m/s, then find the constant height at which the aeroplane is flying.

Solution Let A be the point of observation.

Let E and D be positions of the aeroplane initially and after 15 seconds respectively.

Let BE and CD denote the constant height at which the aeroplane is flying.

Given that $\angle DAC = 30^\circ$, $\angle EAB = 60^\circ$.

Let $BE = CD = h$ metres.

Let $AB = x$ metres.

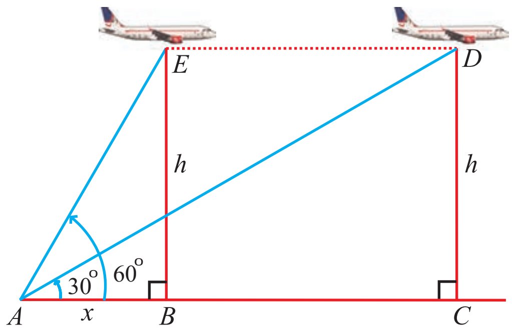


Fig. 7.19

The distance covered in 15 seconds,

$$ED = 200 \times 15 = 3000\text{ m} \qquad \text{(distance travelled = speed} \times \text{time)}$$

Thus, $\quad BC = 3000$ m.

In the right angled $\triangle DAC$,

$$\tan 30^\circ = \frac{CD}{AC}$$

$$\implies \quad CD = AC \tan 30^\circ$$

Thus, $$h = (x + 3000)\frac{1}{\sqrt{3}}. \qquad (1)$$

In the right angled $\triangle EAB$,

$$\tan 60^\circ = \frac{BE}{AB}$$

$$\implies \quad BE = AB \tan 60^\circ \implies h = \sqrt{3}\,x \qquad (2)$$

From (1) and (2), we have $\sqrt{3}\,x = \frac{1}{\sqrt{3}}(x + 3000)$

$$\implies \quad 3x = x + 3000 \quad \implies \quad x = 1500\text{ m}.$$

Thus, from (2) it follows that $h = 1500\sqrt{3}$ m.

The constant height at which the aeroplane is flying, is $1500\sqrt{3}$ m.

Exercise 7.2

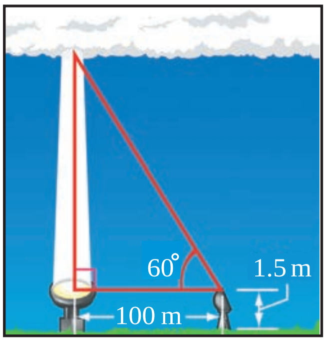
1. A ramp for unloading a moving truck, has an angle of elevation of 30°. If the top of the ramp is 0.9 m above the ground level, then find the length of the ramp.

2. A girl of height 150 cm stands in front of a lamp-post and casts a shadow of length $150\sqrt{3}$ cm on the ground. Find the angle of elevation of the top of the lamp-post .

3. Suppose two insects A and B can hear each other up to a range of 2 m.. The insect A is on the ground 1 m away from a wall and sees her friend B on the wall, about to be eaten by a spider. If A sounds a warning to B and if the angle of elevation of B from A is 30°, will the spider have a meal or not ? (Assume that B escapes if she hears A calling)

4. To find the cloud ceiling, one night an observer directed a spotlight vertically at the clouds. Using a theodolite placed 100 m from the spotlight and 1.5 m above the ground, he found the angle of elevation to be 60°. How high was the cloud ceiling? (Hint : See figure)

   (**Note:** Cloud ceiling is the lowest altitude at which solid cloud is present. The cloud ceiling at airports must be sufficiently high for safe take offs and landings. At night the cloud ceiling can be determined by illuminating the base of the clouds by a spotlight pointing vertically upward.)



5. A simple pendulum of length 40 cm subtends 60° at the vertex in one full oscillation. What will be the shortest distance between the initial position and the final position of the bob? (between the extreme ends)

6. Two crows A and B are sitting at a height of 15 m and 10 m in two different trees vertically opposite to each other . They view a vadai (an eatable) on the ground at an angle of depression 45° and 60° respectively. They start at the same time and fly at the same speed along the shortest path to pick up the vadai. Which bird will succeed in it? Hint : (foot of two trees and vadai (an eatable) are in a straight line)

7. A lamp-post stands at the centre of a circular park. Let P and Q be two points on the boundary such that PQ subtends an angle 90° at the foot of the lamp-post and the angle of elevation of the top of the lamp post from P is 30°. If $PQ = 30$ m, then find the height of the lamp post.

8. A person in an helicopter flying at a height of 700 m, observes two objects lying opposite to each other on either bank of a river. The angles of depression of the objects are 30° and 45°. Find the width of the river. ($\sqrt{3} = 1.732$)

9. A person X standing on a horizontal plane, observes a bird flying at a distance of 100 m from him at an angle of elevation of 30°. Another person Y standing on the roof of a 20 m high building, observes the bird at the same time at an angle of elevation of 45°. If X and Y are on the opposite sides of the bird, then find the distance of the bird from Y.

10. A student sitting in a classroom sees a picture on the black board at a height of 1.5 m from the horizontal level of sight. The angle of elevation of the picture is 30°. As the picture is not clear to him, he moves straight towards the black board and sees the picture at an angle of elevation of 45°. Find the distance moved by the student.

11. A boy is standing at some distance from a 30 m tall building and his eye level from the ground is 1.5 m. The angle of elevation from his eyes to the top of the building increases from 30° to 60° as he walks towards the building. Find the distance he walked towards the building.

12. From the top of a lighthouse of height 200 feet, the lighthouse keeper observes a Yacht and a Barge along the same line of sight . The angles of depression for the Yacht and the Barge are 45° and 30° respectively. For safety purposes the two sea vessels should be atleast 300 feet apart. If they are less than 300 feet , the keeper has to sound the alarm. Does the keeper have to sound the alarm ?

13. A boy standing on the ground, spots a balloon moving with the wind in a horizontal line at a constant height . The angle of elevation of the balloon from the boy at an instant is 60°. After 2 minutes, from the same point of observation,the angle of elevation reduces to 30°. If the speed of wind is $29\sqrt{3}$ m/min. then, find the height of the balloon from the ground level.

14. A straight highway leads to the foot of a tower . A man standing on the top of the tower spots a van at an angle of depression of 30°. The van is approaching the tower with a uniform speed. After 6 minutes, the angle of depression of the van is found to be 60°. How many more minutes will it take for the van to reach the tower?

15. The angles of elevation of an artificial earth satellite is measured from two earth stations, situated on the same side of the satellite, are found to be 30° and 60°. The two earth stations and the satellite are in the same vertical plane. If the distance between the earth stations is 4000 km, find the distance between the satellite and earth. ($\sqrt{3} = 1.732$)

16. From the top of a tower of height 60 m, the angles of depression of the top and the bottom of a building are observed to be 30° and 60° respectively. Find the height of the building.

17. From the top and foot of a 40 m high tower, the angles of elevation of the top of a lighthouse are found to be 30° and 60° respectively. Find the height of the lighthouse. Also find the distance of the top of the lighthouse from the foot of the tower.

18. The angle of elevation of a hovering helicopter as seen from a point 45 m above a lake is 30° and the angle of depression of its reflection in the lake, as seen from the same point and at the same time, is 60°. Find the distance of the helicopter from the surface of the lake.

Exercise 7.3

Choose the correct answer

1. $(1-\sin^2\theta)\sec^2\theta =$

(A) 0 (B) 1 (C) $\tan^2\theta$ (D) $\cos^2\theta$

2. $(1+\tan^2\theta)\sin^2\theta =$

(A) $\sin^2\theta$ (B) $\cos^2\theta$ (C) $\tan^2\theta$ (D) $\cot^2\theta$

3. $(1-\cos^2\theta)(1+\cot^2\theta) =$

(A) $\sin^2\theta$ (B) 0 (C) 1 (D) $\tan^2\theta$

4. $\sin(90^\circ-\theta)\cos\theta+\cos(90^\circ-\theta)\sin\theta =$

(A) 1 (B) 0 (C) 2 (D) –1

5. $1-\dfrac{\sin^2\theta}{1+\cos\theta} =$

(A) $\cos\theta$ (B) $\tan\theta$ (C) $\cot\theta$ (D) $\operatorname{cosec}\theta$

6. $\cos^4 x-\sin^4 x =$

(A) $2\sin^2 x-1$ (B) $2\cos^2 x-1$ (C) $1+2\sin^2 x$ (D) $1-2\cos^2 x$.

7. If $\tan\theta=\dfrac{a}{x}$, then the value of $\dfrac{x}{\sqrt{a^2+x^2}} =$

(A) $\cos\theta$ (B) $\sin\theta$ (C) $\operatorname{cosec}\theta$ (D) $\sec\theta$

8. If $x=a\sec\theta,\ y=b\tan\theta$, then the value of $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2} =$

(A) 1 (B) –1 (C) $\tan^2\theta$ (D) $\operatorname{cosec}^2\theta$

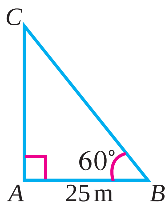
9. $\dfrac{\sec\theta}{\cot\theta+\tan\theta} =$

(A) $\cot\theta$ (B) $\tan\theta$ (C) $\sin\theta$ (D) $-\cot\theta$

10. $\dfrac{\sin(90^\circ-\theta)\sin\theta}{\tan\theta}+\dfrac{\cos(90^\circ-\theta)\cos\theta}{\cot\theta} =$

(A) $\tan\theta$ (B) 1 (C) –1 (D) $\sin\theta$

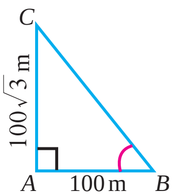
11. In the adjoining figure, $AC =$



(A) 25 m (B) $25\sqrt{3}$ m

(C) $\dfrac{25}{\sqrt{3}}$ m (D) $25\sqrt{2}$ m

12. In the adjoining figure $\angle ABC =$



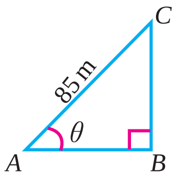
(A) 45° (B) 30°

(C) 60° (D) 50°

13. A man is 28.5 m away from a tower. His eye level above the ground is 1.5 m. The angle of elevation of the tower from his eyes is 45°. Then the height of the tower is

(A) 30 m (B) 27.5 m (C) 28.5 m (D) 27 m

14. In the adjoining figure, $\sin\theta = \frac{15}{17}$. Then BC =



(A) 85 m (B) 65 m

(C) 95 m (D) 75 m

15. $(1+\tan^2\theta)(1-\sin\theta)(1+\sin\theta) =$

(A) $\cos^2\theta - \sin^2\theta$ (B) $\sin^2\theta - \cos^2\theta$

(C) $\sin^2\theta + \cos^2\theta$ (D) 0

16. $(1+\cot^2\theta)(1-\cos\theta)(1+\cos\theta) =$

(A) $\tan^2\theta - \sec^2\theta$ (B) $\sin^2\theta - \cos^2\theta$

(C) $\sec^2\theta - \tan^2\theta$ (D) $\cos^2\theta - \sin^2\theta$

17. $(\cos^2\theta - 1)(\cot^2\theta + 1) + 1 =$

(A) 1 (B) –1 (C) 2 (D) 0

18. $\dfrac{1+\tan^2\theta}{1+\cot^2\theta} =$

(A) $\cos^2\theta$ (B) $\tan^2\theta$ (C) $\sin^2\theta$ (D) $\cot^2\theta$

19. $\sin^2\theta + \dfrac{1}{1+\tan^2\theta} =$

(A) $\operatorname{cosec}^2\theta + \cot^2\theta$ (B) $\operatorname{cosec}^2\theta - \cot^2\theta$

(C) $\cot^2\theta - \operatorname{cosec}^2\theta$ (D) $\sin^2\theta - \cos^2\theta$

20. $9\tan^2\theta - 9\sec^2\theta =$

(A) 1 (B) 0 (C) 9 (D) –9

Do you know?

Paul Erdos (26th March, 1913 – 20th September, 1996) was a Hungarian Mathematician. Erdos was one of the most prolific publishers of research articles in mathematical history, comparable only with Leonhard Euler. He wrote around 1,475 mathematical articles in his life lifetime, while Euler credited with approximately 800 research articles. He strongly believed in and practised mathematics as a social activity, having 511 different collaborators in his lifetime.

8

- Introduction
- Surface area and volume
  - Cylinder
  - Cone
  - Sphere
- Combined figures and invariant volumes

Archimedes

(287 BC - 212 BC)

Greece

Archimedes is remembered as the greatest mathematician of the ancient era.

He contributed significantly in geometry regarding the areas of plane figures and the areas as well as volumes of curved surfaces.

MENSURATION

Measure what is measurable, and make measurable what is not so

-Galileo Galilei

8.1 Introduction

The part of geometry which deals with measurement of lengths of lines, perimeters and areas of plane figures and surface areas and volumes of solid objects is called "Mensuration". The study of measurement of objects is essential because of its uses in many aspects of every day life. In elementary geometry one considers plane, multifaced surfaces as well as certain curved surfaces of solids (for example spheres).

"Surface Area to Volume" ratio has been widely acknowledged as one of the big ideas of Nanoscience as it lays the foundation for understanding size dependent properties that characterise Nanoscience scale and technology.

In this chapter, we shall learn how to find surface areas and volumes of solid objects such as cylinder, cone, sphere and combined objects

8.2 Surface Area

Archimedes of Syracuse, Sicily was a Greek Mathematician who proved that of the volume of a sphere is equal to two-thirds the volume of a circumscribed cylinder. He regarded this as his most vital achievement. He used the method of exhaustion to calculate the area under the arc of a parabola.

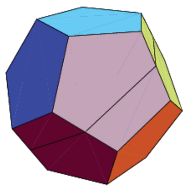


Fig. 8.1

Surface area is the measurement of exposed area of a solid object. Thus, the surface area is the area of all outside surfaces of a 3-dimensional object. The adjoining figures illustrate surface areas of some solids.

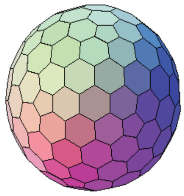


Fig. 8.2

8.2.1 Right Circular Cylinder

If we take a number of circular sheets of paper or cardboard of the same shape and size and stack them up in a vertical pile, then by this process, we shall obtain a solid object known as a Right Circular Cylinder. Note that it has been kept at right angles to the base, and the base is circular. (See Fig. 8.3)

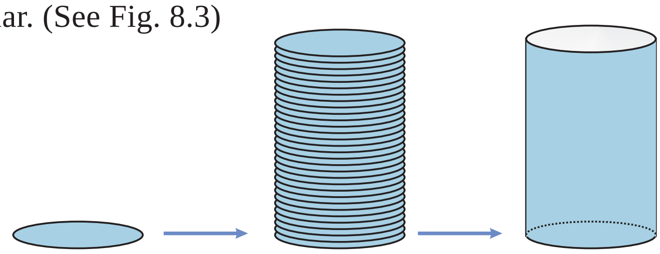


Fig. 8.3

Definition

If a rectangle revolves about its one side and completes a full rotation, the solid thus formed is called a right circular cylinder.

Activity

Let $ABCD$ be a rectangle. Assume that it revolves about its side AB and completes a full rotation. This revolution generates a right circular cylinder as shown in the figures. AB is called the axis of the cylinder. The length AB is the length or the height of the cylinder and AD or BC is called its radius.

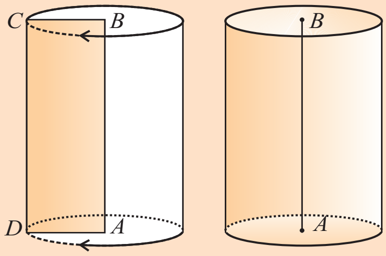


Fig. 8.4

Note

(i) If the base of a cylinder is not circular then it is called oblique cylinder.

(ii) If the base is circular but not perpendicular to the axis of the cylinder, then the cylinder is called circular cylinder.

(iii) If the axis is perpendicular to the circular base, then the cylinder is called right circular cylinder.

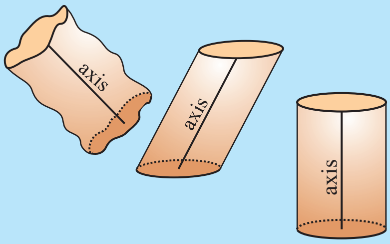


Fig. 8.5

(i) Curved Surface area of a solid right circular cylinder

In the adjoining figure, the bottom and top face of the right circular cylinder are concurrent circular regions, parallel to each other. The vertical surface of the cylinder is curved and hence its area is called the curved surface or lateral surface area of the cylinder.

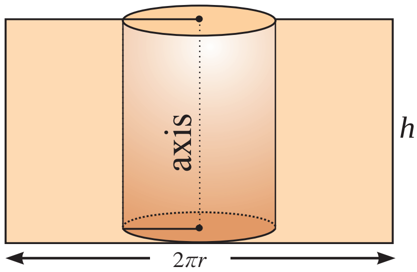


Fig. 8.6

Curved Surface Area of a cylinder, CSA = Circumference of the base $\times$ Height $= 2\pi r \times h$

$= 2\pi rh$ sq. units.

(ii) Total Surface Area of a solid right circular cylinder

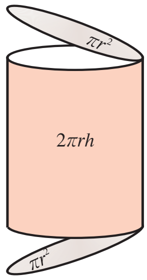


Fig. 8.7

Total Surface Area, TSA = Area of the Curved Surface Area + $2 \times$ Base Area

$= 2\pi rh + 2 \times \pi r^2$

Thus, TSA $= 2\pi r(h + r)$ sq.units.

(iii) Right circular hollow cylinder

Solids like iron pipe, rubber tube, etc., are in the shape of hollow cylinders. For a hollow cylinder of height h with external and internal radii R and r respectively,

we have, curved surface area, CSA = External surface area + Internal surface area

$= 2\pi Rh + 2\pi rh$

Thus, CSA $= 2\pi h(R + r)$ sq.units

Total surface area, TSA = CSA + $2 \times$ Base area

$= 2\pi h(R + r) + 2 \times [\pi R^2 - \pi r^2]$

$= 2\pi h(R + r) + 2\pi(R + r)(R - r)$

$\therefore$ TSA $= 2\pi(R + r)(R - r + h)$ sq.units.

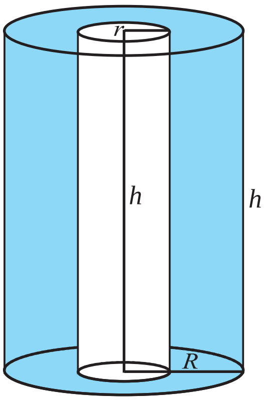


Fig. 8.8

Remark

Thickness of the hollow cylinder, w $= R - r$.

Note

In this chapter, for π we take an approximate value $\frac{22}{7}$ whenever it is required.

Example 8.1

A solid right circular cylinder has radius 7cm and height 20cm. Find its (i) curved surface area and (ii) total surface area. (Take $\pi = \frac{22}{7}$)

Solution Let r and h be the radius and height of the solid right circular cylinder respectively.

Given that r = 7cm and h = 20cm

Curved surface area, CSA $= 2\pi rh$

$= 2 \times \frac{22}{7} \times 7 \times 20$

Thus, the curved surface area = 880 sq.cm

Now, the total surface area $= 2\pi r(h + r)$

$= 2 \times \frac{22}{7} \times 7 \times [20 + 7] = 44 \times 27$

Thus, the total surface area = 1188 sq.cm.

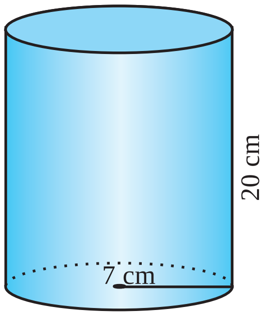


Fig. 8.9

Example 8.2

If the total surface area of a solid right circular cylinder is 880 sq.cm and its radius is 10 cm, find its curved surface area. (Take $\pi = \frac{22}{7}$)

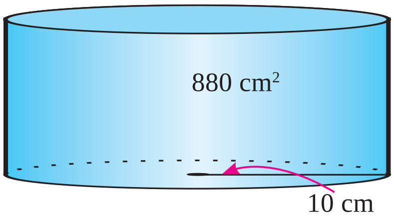


Fig. 8.10

Solution Let r and h be the radius and height of the solid right circular cylinder respectively.

Let S be the total surface area of the solid right circular cylinder.

Given that $r = 10$ cm and $S = 880$ cm^2

$$\text{Now, } S = 880 \implies 2\pi r[h + r] = 880$$

$$\implies 2\times\frac{22}{7}\times 10[h + 10] = 880$$

$$\implies h + 10 = \frac{880\times 7}{2\times 22\times 10}$$

$$\implies h + 10 = 14$$

Thus, the height of the cylinder, $h = 4$ cm

Now, the curved surface area, CSA is

$$2\pi rh = 2\times\frac{22}{7}\times 10\times 4 = \frac{1760}{7}$$

Thus, the curved surface area of the cylinder $= 251\frac{3}{7}$ sq.cm.

Aliter :

$$\text{CSA} = \text{TSA} - 2\times \text{Area of the base}$$
$$= 880 - 2\times\pi r^2$$
$$= 880 - 2\times\frac{22}{7}\times 10^2$$
$$= \frac{1760}{7} = 251\frac{3}{7} \text{ sq.cm.}$$

Example 8.3

The ratio between the base radius and the height of a solid right circular cylinder is 2 : 5. If its curved surface area is $\frac{3960}{7}$ sq.cm, find the height and radius. (use $\pi = \frac{22}{7}$)

Solution Let r and h be the radius and height of the right circular cylinder respectively.

Given that $r : h = 2 : 5 \implies \frac{r}{h} = \frac{2}{5}$. Thus, $r = \frac{2}{5}h$

Now, the curved surface area, $\text{CSA} = 2\pi rh$

$$\implies 2\times\frac{22}{7}\times\frac{2}{5}\times h\times h = \frac{3960}{7}$$

$$\implies h^2 = \frac{3960\times 7\times 5}{2\times 22\times 2\times 7} = 225$$

Thus, $h = 15 \implies r = \frac{2}{5}h = 6.$

Hence, the height of the cylinder is 15 cm and the radius is 6 cm.

Example 8.4

The diameter of a road roller of length 120 cm is 84 cm. If it takes 500 complete revolutions to level a playground, then find the cost of levelling it at the cost of 75 paise per square metre. (Take $\pi = \frac{22}{7}$)

Solution Given that $r = 42$ cm, $h = 120$ cm

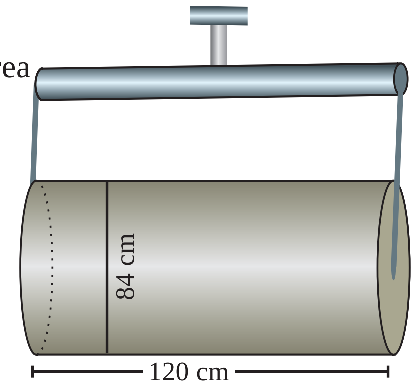


Fig. 8.11

$$\left.\begin{array}{r}\text{Area covered by the roller}\\ \text{in one revolution}\end{array}\right\} = \left\{\begin{array}{l}\text{Curved Surface Area}\\ \text{of the road roller.}\end{array}\right.$$

$$= 2\pi rh$$

$$= 2 \times \frac{22}{7} \times 42 \times 120$$

$$= 31680\,\text{cm}^2.$$

$$\left.\begin{array}{r}\text{Area covered by the}\\ \text{roller in 500 revolutions}\end{array}\right\} = 31680 \times 500$$

$$= 15840000\,\text{cm}^2$$

$$= \frac{15840000}{10000} = 1584\,\text{m}^2 \qquad (10{,}000\ \text{cm}^2 = 1\ \text{sq.m})$$

Cost of levelling per 1sq.m. $= ₹\,\frac{75}{100}$

Thus, cost of levelling the play ground $= \frac{1584 \times 75}{100} = ₹\,1188.$

Example 8.5

The internal and external radii of a hollow cylinder are 12 cm and 18 cm respectively. If its height is 14cm, then find its curved surface area and total surface area. (Take $\pi = \frac{22}{7}$)

Solution Let r, R and h be the internal and external radii and the height of a hollow cylinder respectively.

Given that $r = 12$ cm, $R = 18$ cm, $h = 14$ cm

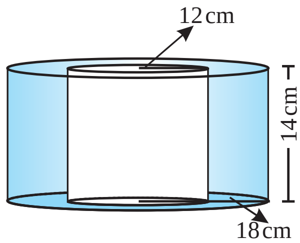


Fig. 8.12

Now, curved surface area, CSA $= 2\pi h(R+r)$

Thus, CSA $= 2 \times \frac{22}{7} \times 14 \times (18+12)$

$= 2640$ sq.cm

Total surface area, TSA $= 2\pi(R+r)(R-r+h)$

$$= 2 \times \frac{22}{7} \times (18+12)(18-12+14)$$

$$= 2 \times \frac{22}{7} \times 30 \times 20 = \frac{26400}{7}.$$

Thus, the total surface area $= 3771\frac{3}{7}$ sq.cm.

8.2.2 Right Circular Cone

In our daily life we come across many solids or objects like ice cream container, the top of the temple car, the cap of a clown in a circus, the mehandi container. Mostly the objects mentioned above are in the shape of a right circular cone.

A cone is a solid object that tapers smoothly from a flat base to a point called vertex. In general, the base may not be of circular shape. Here, cones are assumed to be right circular, where right means that the axis that passes through the centre of the base is at right angles to its plane, and circular means that the base is a circle. In this section, let us define a right circular cone and find its surface area. One can visualise a cone through the following activity.

Activity

Take a thick paper and cut a right angled $\triangle ABC$, right angled at B. Paste a long thick string along one of the perpendicular sides say AB of the triangle. Hold the string with your hands on either side of the triangle and rotate the triangle about the string.

What happens? Can you recognize the shape formed on the rotation of the triangle around the string?. The shape so formed is a right circular cone.

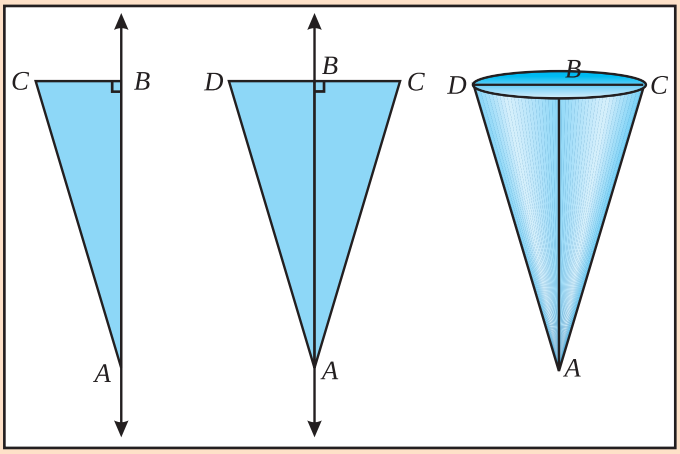


Fig. 8.13

If a right angled $\triangle ABC$ is revolved 360° about the side AB containing the right angle, the solid thus formed is called a right circular cone.

The length AB is called the height of the cone.

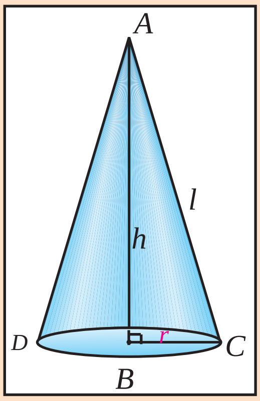


Fig. 8.14

The length BC is called the radius of its base ($BC = r$). The length AC is called the slant height l of the cone ($AC = AD = l$).

In the right angled $\triangle ABC$

We have, $\quad l = \sqrt{h^2 + r^2}\quad$ (Pythagoras theorem)

$$h = \sqrt{l^2 - r^2}$$

$$r = \sqrt{l^2 - h^2}$$

Note

(i) If the base of a cone is not circular then, it is called **oblique cone.**

(ii) If the circular base is not perpendicular to the axis then, it is called **circular cone.**

(iii) If the vertex is directly above the centre of the circular base then, it is a **right circular cone.**

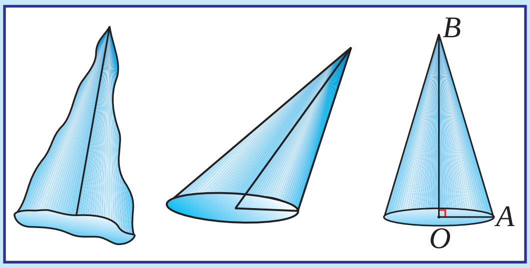


Fig. 8.15

(i) Curved surface area of a hollow cone

Let us consider a sector with radius l and central angle θ°. Let L denote the length of the arc. Thus, $\dfrac{2\pi l}{L} = \dfrac{360^\circ}{\theta^\circ}$

$$\Longrightarrow \quad L = 2\pi l \times \frac{\theta^\circ}{360^\circ} \qquad (1)$$

Now, join the radii of the sector to obtain a right circular cone.

Let r be the radius of the cone.

Hence, $\quad L = 2\pi r$

From (1) we obtain,

$$2\pi r = 2\pi l \times \frac{\theta^\circ}{360^\circ}$$

$$\Longrightarrow \quad r = l\left(\frac{\theta^\circ}{360^\circ}\right)$$

$$\Longrightarrow \quad \frac{r}{l} = \left(\frac{\theta^\circ}{360^\circ}\right)$$

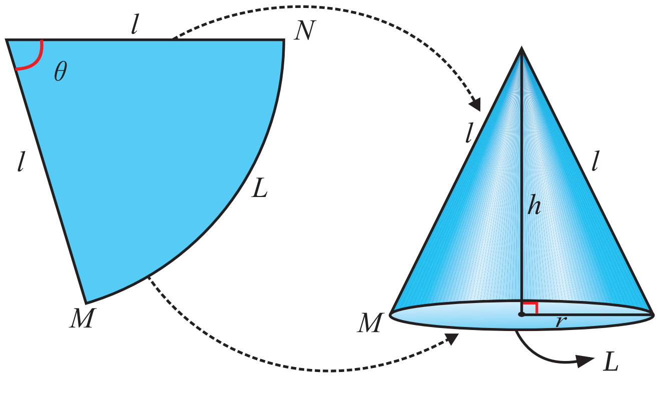


Fig. 8.16

Remarks

When a sector of a circle is folded into a cone, the following conversions are taking place:

Sector	Cone
Radius (l) →	Slant height (l)
Arc Length (L) →	Perimeter of the base $2\pi r$
Area →	Curved Surface Area πrl

Let A be the area of the sector. Then

$$\frac{\pi l^2}{A} = \frac{360^\circ}{\theta^\circ} \qquad (2)$$

Then the curved surface area of the cone $\Big\}$ = Area of the sector

Thus, the area of the curved surface of the cone $\Big\}$ $A = \pi l^2\left(\dfrac{\theta^\circ}{360^\circ}\right) = \pi l^2\left(\dfrac{r}{l}\right)$.

Hence, the curved surface area of the cone $= \pi rl$ sq.units.

(ii) Total surface area of the solid right circular cone

Total surface area of the solid cone = $\Big\{$ Curved surface area of the cone + Area of the base

$$= \pi rl + \pi r^2$$

Total surface area of the solid cone $= \pi r(l + r)$ sq.units.

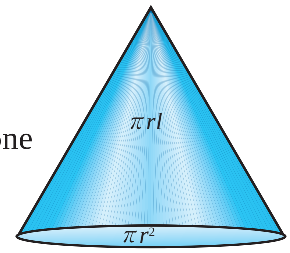


Fig. 8.17

Example 8.6

Radius and slant height of a solid right circular cone are 35cm and 37cm respectively. Find the curved surface area and total surface area of the cone. (Take $\pi = \dfrac{22}{7}$)

Solution Let r and l be the radius and the slant height of the solid right circular cone respectively.

$$r = 35 \text{ cm}, \; l = 37 \text{ cm}$$

Curved surface area, $\text{CSA} = \pi r l = \pi(35)(37)$

$\text{CSA} = 4070$ sq.cm

Total surface area, $\text{TSA} = \pi r[l + r]$

$$= \frac{22}{7} \times 35 \times [37 + 35]$$

Thus, $\text{TSA} = 7920$ sq.cm.

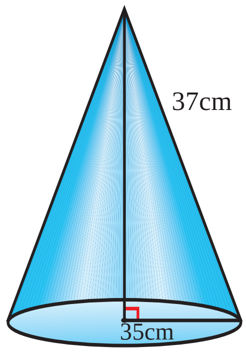


Fig. 8.18

Example 8.7

Let O and C be the centre of the base and the vertex of a right circular cone. Let B be any point on the circumference of the base. If the radius of the cone is 6 cm and if $\angle OBC = 60^o$, then find the height and curved surface area of the cone.

Solution Given that radius OB = 6 cm and $\angle OBC = 60^o$.

In the right angled $\triangle OBC$,

$$\cos 60^o = \frac{OB}{BC}$$

$$\Longrightarrow \quad BC = \frac{OB}{\cos 60^\circ}$$

$$\therefore \quad BC = \frac{6}{\left(\frac{1}{2}\right)} = 12 \text{ cm}$$

Thus, the slant height of the cone, $l = 12$ cm

In the right angled $\triangle OBC$, we have

$$\tan 60^o = \frac{OC}{OB}$$

$$\Longrightarrow \quad OC = OB \tan 60^\circ = 6\sqrt{3}$$

Thus, the height of the cone, OC = $6\sqrt{3}$ cm

Now, the curved surface area is $\pi r l = \pi \times 6 \times 12 = 72\pi \text{ cm}^2$.

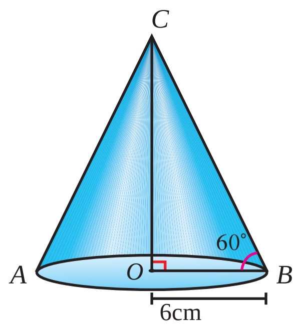


Fig. 8.19

Example 8.8

A sector containing an angle of 120° is cut off from a circle of radius 21 cm and folded into a cone. Find the curved surface area of the cone. (Take $\pi = \frac{22}{7}$)

Solution Let r be the base radius of the cone.

Angle of the sector, $\theta = 120^\circ$

Radius of the sector, $R = 21$ cm

When the sector is folded into a right circular cone, we have

circumference of the base of the cone

= Length of the arc

$$\implies \quad 2\pi r = \frac{\theta}{360^\circ}\times 2\pi R$$

$$\implies \quad r = \frac{\theta}{360^\circ}\times R$$

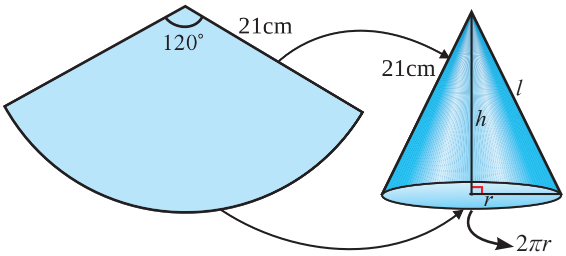


Fig. 8.20

Thus, the base radius of the cone, $r = \frac{120^\circ}{360^\circ}\times 21 = 7$ cm.

Also, the slant height of the cone ,

l = Radius of the sector

Thus, $l = R \implies l = 21$ cm.

Now , the curved surface area of the cone,

$$\text{CSA} = \pi r l$$

$$= \frac{22}{7}\times 7\times 21 = 462.$$

Thus, the curved surface area of the cone is 462 sq.cm.

Aliter :

CSA of the cone = Area of the sector

$$= \frac{\theta^\circ}{360^\circ}\times\pi\times R^2$$

$$= \frac{120}{360}\times\frac{22}{7}\times 21\times 21$$

$$= 462 \text{ sq.cm.}$$

8.2.3 Sphere

If a circular disc is rotated about one of its diameter, the solid thus generated is called sphere. Thus sphere is a 3- dimensional object which has surface area and volume.

(i) Curved surface area of a solid sphere

Activity

Take a circular disc, paste a string along a diameter of the disc and rotate it 360°. The object so created looks like a ball. The new solid is called sphere.

The following activity may help us to visualise the surface area of a sphere as four times the area of the circle with the same radius.

- Take a plastic ball.
- Fix a pin at the top of the ball.
- Wind a uniform thread over the ball so as to cover the whole curved surface area.
- Unwind the thread and measure the length of the thread used.
- Cut the thread into four equal parts.
- Place the strings as shown in the figures.

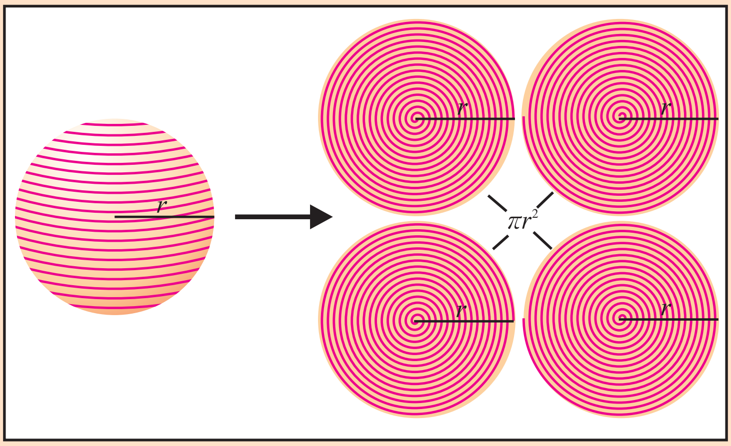


Fig. 8.21

- Measure the radius of the sphere and the circles formed.

Now, the radius of the sphere = radius of the four equal circles.

Thus, curved surface area of the sphere, CSA = $4\times$ Area of the circle = $4\times\pi r^2$

$\therefore$ The curved surface area of a sphere = $4\pi r^2$ sq. units.

(ii) Solid hemisphere

A plane passing through the centre of a solid sphere divides the sphere into two equal parts. Each part of the sphere is called a solid hemisphere.

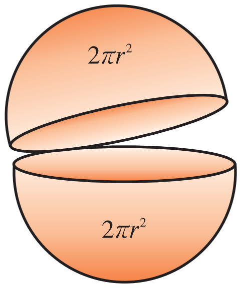


Fig. 8.22

$$\text{Curved surface area of a hemisphere} = \frac{\text{CSA of the Sphere}}{2}$$

$$= \frac{4\pi r^2}{2} = 2\pi r^2 \text{ sq.units.}$$

Total surface area of a hemisphere, TSA = Curved Surface Area + Area of the base Circle

$$= 2\pi r^2 + \pi r^2$$

$$= 3\pi r^2 \text{ sq.units.}$$

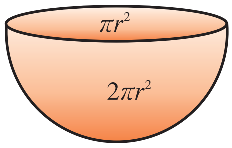


Fig. 8.23

(iii) Hollow hemisphere

Let R and r be the outer and inner radii of the hollow hemisphere.

Now, its curved surface area = Outer surface area + Inner surface area

$$= 2\pi R^2 + 2\pi r^2$$

$$= 2\pi(R^2 + r^2) \text{ sq.units}.$$

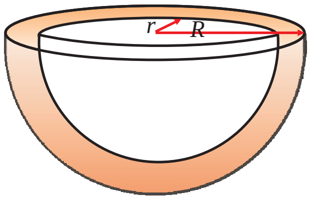


Fig. 8.24

The total surface area = $\begin{cases} \text{Outer surface area + Inner surface area} \\ + \text{ Area at the base} \end{cases}$

$$= 2\pi R^2 + 2\pi r^2 + \pi(R^2 - r^2)$$

$$= 2\pi(R^2 + r^2) + \pi(R + r)(R - r) \text{ sq.units.}$$

$$= \pi(3R^2 + r^2) \text{ sq. units}$$

Example 8.9

A hollow sphere in which a circus motorcyclist performs his stunts, has an inner diameter of 7 m. Find the area available to the motorcyclist for riding. (Take $\pi = \frac{22}{7}$)

Solution Inner diameter of the hollow sphere, $2r = 7$ m.

Available area to the motorcyclist for riding = Inner surface area of the sphere

$$= 4\pi r^2 = \pi(2r)^2$$

$$= \frac{22}{7} \times 7^2$$

Available area to the motorcyclist for riding = 154 sq.m.

Example 8.10

Total surface area of a solid hemisphere is 675π sq.cm. Find the curved surface area of the solid hemisphere.

Solution Given that the total surface area of the solid hemisphere,

$$3\pi r^2 = 675\pi \text{ sq. cm}$$

$$\Longrightarrow \quad r^2 = 225$$

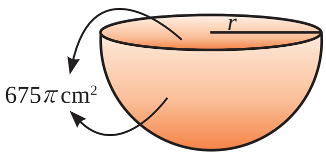


Fig. 8.25

Now, the curved surface area of the solid hemisphere,

$$\text{CSA} = 2\pi r^2 = 2\pi \times 225 = 450\pi \text{ sq.cm.}$$

Example 8.11

The thickness of a hemispherical bowl is 0.25 cm. The inner radius of the bowl is 5 cm. Find the outer curved surface area of the bowl.(Take $\pi = \frac{22}{7}$)

Solution Let *r*, *R* and *w* be the inner and outer radii and thickness of the hemispherical bowl respectively.

Given that $r = 5\text{cm}, w = 0.25\text{cm}$

$\therefore \quad R = r + w = 5 + 0.25 = 5.25\text{cm}$

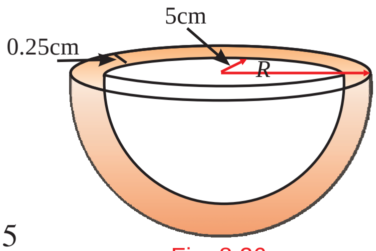


Fig. 8.26

Now, outer surface area of the bowl $= 2\pi R^2$

$$= 2 \times \frac{22}{7} \times 5.25 \times 5.25$$

Thus, the outer surface area of the bowl = 173.25 sq.cm.

Exercise 8.1

1. A solid right circular cylinder has radius of 14 cm and height of 8 cm. Find its curved surface area and total surface area.

2. The total surface area of a solid right circular cylinder is 660 sq.cm. If its diameter of the base is 14 cm, find the height and curved surface area of the cylinder.

3. Curved surface area and circumference at the base of a solid right circular cylinder are 4400 sq.cm and 110 cm respectively. Find its height and diameter.

4. A mansion has 12 right cylindrical pillars each having radius 50 cm and height 3.5 m. Find the cost to paint the lateral surface of the pillars at ₹ 20 per square metre.

5. The total surface area of a solid right circular cylinder is 231 cm^2. Its curved surface area is two thirds of the total surface area. Find the radius and height of the cylinder.

6. The total surface area of a solid right circular cylinder is 1540 cm^2. If the height is four times the radius of the base, then find the height of the cylinder.`

7. The radii of two right circular cylinders are in the ratio of 3 : 2 and their heights are in the ratio 5 : 3. Find the ratio of their curved surface areas.

8. The external surface area of a hollow cylinder is 540π sq.cm. Its internal diameter is 16 cm and height is 15 cm. Find the total surface area.
9. The external diameter of a cylindrical shaped iron pipe is 25 cm and its length is 20 cm. If the thickness of the pipe is 1cm, find the total surface area of the pipe.
10. The radius and height of a right circular solid cone are 7 cm and 24 cm respectively. Find its curved surface area and total surface area.
11. If the vertical angle and the radius of a right circular cone are 60° and 15 cm respectively, then find its height and slant height.
12. If the circumference of the base of a solid right circular cone is 236 cm and its slant height is 12 cm, find its curved surface area.
13. A heap of paddy is in the form of a cone whose diameter is 4.2 m and height is 2.8 m. If the heap is to be covered exactly by a canvas to protect it from rain, then find the area of the canvas needed.
14. The central angle and radius of a sector of a circular disc are 180° and 21 cm respectively. If the edges of the sector are joined together to make a hollow cone, then find the radius of the cone.
15. Radius and slant height of a solid right circular cone are in the ratio 3:5. If the curved surface area is 60π sq.cm, then find its total surface area.
16. If the curved surface area of solid a sphere is 98.56 cm^2, then find the radius of the sphere..
17. If the curved surface area of a solid hemisphere is 2772 sq.cm, then find its total surface area.
18. Radii of two solid hemispheres are in the ratio 3 : 5. Find the ratio of their curved surface areas and the ratio of their total surface areas.
19. Find the curved surface area and total surface area of a hollow hemisphere whose outer and inner radii are 4.2 cm and 2.1 cm respectively.
20. The inner curved surface area of a hemispherical dome of a building needs to be painted. If the circumference of the base is 17.6m, find the cost of painting it at the rate of ₹5 per sq.m.

8.3 Volume

So far we have seen the problems related to the surface area of some solids. Now we shall learn how to calculate volumes of some familiar solids. Volume is literally the 'amount of space filled'. The volume of a solid is a numerical characteristic of the solid.

For example, if a body can be decomposed into finite set of unit cubes (cubes of unit sides), then the volume is equal to the number of these cubes.

The cube in the figure, has a volume

$$= \text{length} \times \text{width} \times \text{height}$$

$$= 1\,\text{cm} \times 1\,\text{cm} \times 1\,\text{cm} = 1\,\text{cm}^3.$$

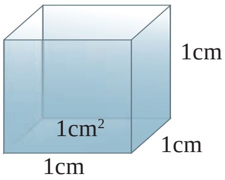


Fig. 8.27

If we say that the volume of an object is 100 cu.cm, then it implies that we need 100 cubes each of 1 cm^3 volume to fill this object completely.

Just like surface area, volume is a positive quantity and is invariant with respect to displacement. Volumes of some solids are illustrated below.

8.3.1 Volume of a right circular cylinder

(i) Volume of a solid right circular cylinder

The volume of a solid right circular cylinder is the product of the base area and height.

That is, the volume of the cylinder, V = Area of the base $\times$ height

$$= \pi r^2 \times h$$

Thus, the volume of a cylinder, $V = \pi r^2 h$ cu. units.

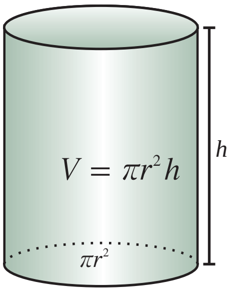


Fig. 8.28

(ii) Volume of a hollow cylinder (Volume of the material used)

Let R and r be the external and internal radii of a hollow right circular cylinder respectively. Let h be its height.

Then, the volume, $V = \left.\begin{matrix}\text{Volume of the}\\ \text{outer cylinder}\end{matrix}\right\} - \left\{\begin{matrix}\text{Volume of the}\\ \text{inner cylinder}\end{matrix}\right.$

$$= \pi R^2 h - \pi r^2 h$$

Hence, the volume of a hollow cylinder,

$$V = \pi h (R^2 - r^2) \text{ cu. units.}$$

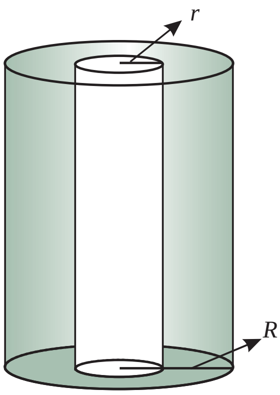


Fig. 8.29

Example 8.12

If the curved surface area of a right circular cylinder is 704 sq.cm, and height is 8 cm, find the volume of the cylinder in litres. (Take $\pi = \frac{22}{7}$)

Solution Let r and h be the radius and height of the right circular cylinder respectively.

Given that $h = 8$ cm and CSA = 704 sq.cm

Now, CSA = 704

$\Longrightarrow \quad 2\pi rh = 704$

$2 \times \frac{22}{7} \times r \times 8 = 704$

$\therefore \quad r = \frac{704 \times 7}{2 \times 22 \times 8} = 14\,\text{cm}$

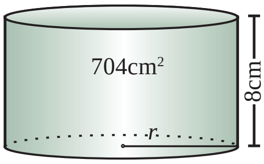


Fig. 8.30

Thus, the volume of the cylinder, $V = \pi r^2 h$

$$= \frac{22}{7} \times 14 \times 14 \times 8$$

$$= 4928 \text{ cu.cm.}$$

Hence, the volume of the cylinder $= 4.928$ litres. (1000 cu.cm = l litre)

Example 8.13

A hollow cylindrical iron pipe is of length 28 cm. Its outer and inner diameters are 8 cm and 6 cm respectively. Find the volume of the pipe and weight of the pipe if 1 cu.cm of iron weighs 7 gm.(Take $\pi = \frac{22}{7}$)

Solution Let r, R and h be the inner, outer radii and height of the hollow cylindrical pipe respectively.

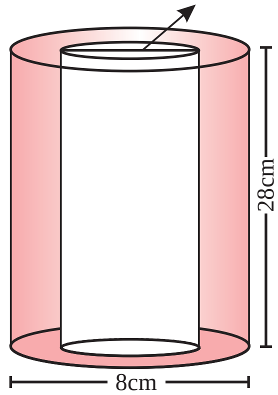


Fig. 8.31

Given that $2r = 6$ cm, $2R = 8$ cm , $h = 28$ cm

Now, the volume of the pipe, $V = \pi \times h \times (R + r)(R - r)$

$$= \frac{22}{7} \times 28 \times (4+3)(4-3)$$

$\therefore$ Volume, $V = 616$ cu. cm

Weight of 1 cu.cm of the metal $= 7$ gm

Weight of the 616 cu. cm of metal $= 7 \times 616$ gm

Thus, the weight of the pipe $= 4.312$ kg.

Example 8.14

Base area and volume of a solid right circular cylinder are 13.86 sq.cm, and 69.3 cu.cm respectively. Find its height and curved surface area.(Take $\pi = \frac{22}{7}$)

Solution Let A and V be the base area and volume of the solid right circular cylinder respectively.

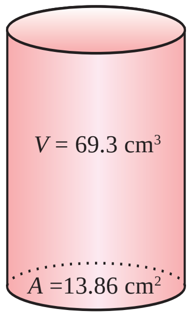


Fig. 8.32

Given that the base area, $A = \pi r^2 = 13.86$ sq.cm and

volume, $V = \pi r^2 h = 69.3$ cu.cm.

Thus, $\pi r^2 h = 69.3$

$\Longrightarrow$ $13.86 \times h = 69.3$

$\therefore$ $h = \frac{69.3}{13.86} = 5$ cm.

Now, the base area $= \pi r^2 = 13.86$

$$\frac{22}{7} \times r^2 = 13.86$$

$$r^2 = 13.86 \times \frac{7}{22} = 4.41 \quad \Longrightarrow \quad r = \sqrt{4.41} = 2.1 \text{ cm.}$$

Now, Curved surface area, CSA $= 2\pi rh$

$$= 2\times\frac{22}{7}\times 2.1\times 5$$

Thus, CSA $= 66$ sq.cm.

8.3.2 Volume of a right circular cone

Let r and h be the base radius and the height of a right circular cone respectively.

The volume V of the cone is given by the formula: $V=\frac{1}{3}\times\pi r^2 h$ cu. units. To justify this formula, let us perform the following activity.

Activity

Make a hollow cone and a hollow cylinder like in the figure given below with the same height and same radius.Now, practically we can find out the volume of the cone by doing the process given below. Fill the cone with sand or liquid and then pour it into the cylinder. Continuing this experiment, we see that the cylinder will be filled completely by sand / liquid at the third time.

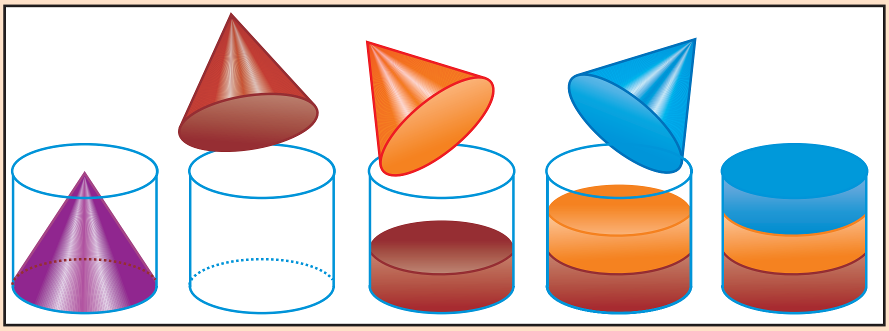


Fig. 8.33

From this simple activity, if r and h are the radius and height of the cylinder, then we find that $3\times$(Volume of the cone) = Volume of the cylinder $= \pi r^2 h$

Thus, the volume of the cone $= \frac{1}{3}\times\pi r^2 h$ cu. units.

Example 8.15

The volume of a solid right circular cone is 4928 cu. cm. If its height is 24 cm, then find the radius of the cone. (Take $\pi = \frac{22}{7}$)

Solution Let r, h and V be the radius, height and volume of a solid cone respectively.

Given that $V =$ 4928 cu.cm and $h = 24$ cm

Thus, we have $\frac{1}{3}\pi r^2 h = 4928$

$$\Longrightarrow \frac{1}{3}\times\frac{22}{7}\times r^2\times 24 = 4928$$

$$\Longrightarrow r^2 = \frac{4928\times 3\times 7}{22\times 24} = 196.$$

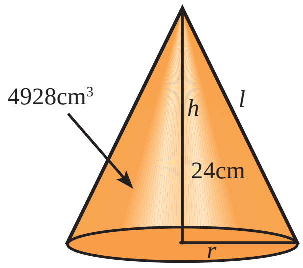


Fig. 8.34

Thus, the base radius of the cone, $r = \sqrt{196} = 14$ cm.

8.3.3 Volume of a Frustum of a Cone

Let us consider a right circular solid cone and cut it into two solids so as to obtain a smaller right circular cone. The other portion of the cone is called frustum of the cone. This is illustrated in the following activity.

Activity

Take some clay and form a right circular cone. Cut it with a knife parallel to its base. Remove the smaller cone. What are you left with? The left out portion of the solid cone is called frustum of the cone. The Latin word frustum means "piece cut off" and its plural is frusta.

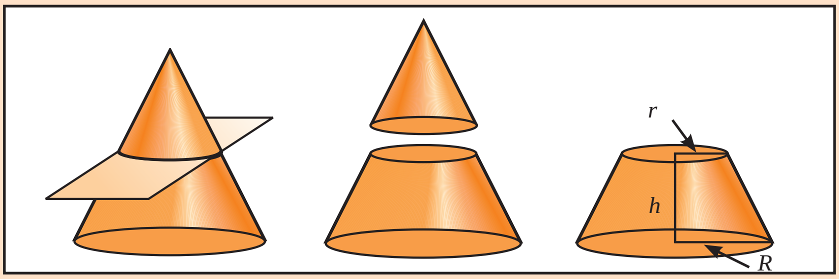


Fig. 8.35

Hence, if a solid right circular cone is sliced with a plane parallel to its base , the part of the cone containing the base is called a frustum of the cone. Thus a frustum has two circular discs, one at the bottom and the other at the top of it.

Let us find the volume of a frustum of a cone.

The volume of a frustum of a cone is nothing but the difference between volumes of two right circular cones. (See Fig. 8.35) Consider a frustum of a solid right circular cone.

Let R be the radius of the given cone. Let r and x be the radius and the height of the smaller cone obtained after removal of the frustum from the given cone.

Let h be the height of the frustum.

$$\text{Now, } \left.\begin{array}{l}\text{the volume of the}\\ \text{frustum of the cone}\end{array}\right\}, \; V = \left.\begin{array}{c}\text{Volume of the}\\ \text{given cone}\end{array}\right\} - \left\{\begin{array}{l}\text{Volume of the}\\ \text{smaller cone}\end{array}\right.$$

$$= \frac{1}{3}\times\pi\times R^2\times(x+h) - \frac{1}{3}\times\pi\times r^2\times x$$

$$\text{Thus, } V = \frac{1}{3}\pi[x(R^2-r^2)+R^2h]. \qquad (1)$$

From the Fig. 8.36 we see that $\Delta BFE \sim \Delta DGE$

$$\therefore \qquad \frac{BF}{DG} = \frac{FE}{GE}$$

$$\Longrightarrow \qquad \frac{R}{r} = \frac{x+h}{x}$$

$$\Longrightarrow \quad Rx - rx = rh$$

$$\Longrightarrow \quad x(R - r) = rh$$

Thus, we get $\quad x = \dfrac{rh}{R - r} \qquad (2)$

Now, (1) $\Longrightarrow \quad V = \frac{1}{3}\pi[x(R^2 - r^2) + R^2 h]$

$$\Longrightarrow \quad = \frac{1}{3}\pi[x(R - r)(R + r) + R^2 h]$$

$$\Longrightarrow \quad = \frac{1}{3}\pi[rh(R + r) + R^2 h] \text{ using (2)}$$

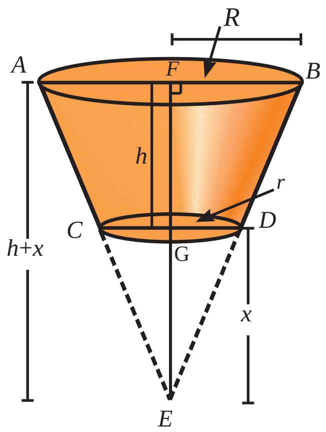


Fig. 8.36

Hence, the volume of the frustum of the cone,

$$V = \frac{1}{3}\pi h(R^2 + r^2 + Rr) \text{ cu. units.}$$

Note

* Curved surface area of a frustum of a cone $= \pi(R + r)l$, where $l = \sqrt{h^2 + (R - r)^2}$

* Total surface area of a frustum of a the cone $= \pi l(R + r) + \pi R^2 + \pi r^2, l = \sqrt{h^2 + (R - r)^2}$

(* Not to be used for examination purpose)

Example 8.16

The radii of two circular ends of a frustum shaped bucket are 15 cm and 8 cm. If its depth is 63 cm, find the capacity of the bucket in litres. (Take $\pi = \frac{22}{7}$)

Solution Let R and r are the radii of the circular ends at the top and bottom and h be the depth of the bucket respectively.

Given that $R = 15$ cm , $r = 8$ cm and $h = 63$ cm.

The volume of the bucket (frustum)

$$= \frac{1}{3}\pi h(R^2 + r^2 + Rr)$$

$$= \frac{1}{3} \times \frac{22}{7} \times 63 \times (15^2 + 8^2 + 15 \times 8)$$

$$= 26994 \text{ cu.cm.}$$

$$= \frac{26994}{1000} \text{ litres} \qquad (1000 \text{ cu.cm} = 1 \text{ litre})$$

Fig. 8.37

Thus, the capacity of the bucket = 26.994 litres.

8.3.4 Volume of a Sphere

(i) Volume of a Solid Sphere

The following simple experiment justifies the formula for volume of a sphere,

$$V = \frac{4}{3}\pi r^3 \text{ cu.units.}$$

Activity

Take a cylindrical shaped container of radius R and height H. Fill the container with water. Immerse a solid sphere of radius r, where $R > r$, in the container and fill the displaced water into another cylindrical shaped container of radius r and height H. The height of the water level is equal to $\frac{4}{3}$ times of its radius ($h = \frac{4}{3}r$). Now, the volume of the solid sphere is same as that of the displaced water.

Volume of the displaced water, V = Base area x Height

$$= \pi r^2 \times \frac{4}{3}r \text{ (here, height of the water level } h = \frac{4}{3}r)$$

$$= \frac{4}{3}\pi r^3$$

Thus, the volume of the sphere, $V = \frac{4}{3}\pi r^3$ cu.units.

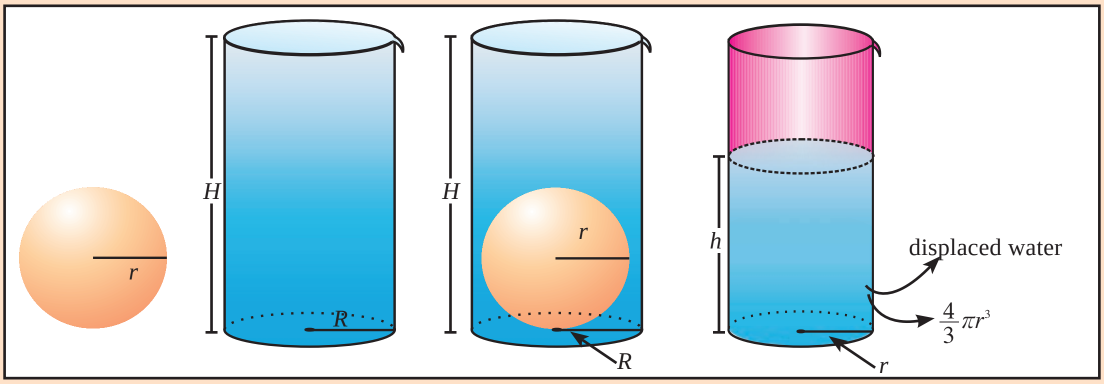


Fig. 8.38

(ii) Volume of a hollow sphere (Volume of the material used)

If the inner and outer radius of a hollow sphere are r and R respectively, then

$$\left.\begin{matrix}\text{Volume of the}\\\text{hollow sphere}\end{matrix}\right\} = \left.\begin{matrix}\text{Volume of the}\\\text{outer sphere}\end{matrix}\right\} - \left\{\begin{matrix}\text{Volume of the}\\\text{inner sphere}\end{matrix}\right.$$

$$= \frac{4}{3}\pi R^3 - \frac{4}{3}\pi r^3$$

$\therefore$ Volume of hollow sphere $= \frac{4}{3}\pi (R^3 - r^3)$ cu. units.

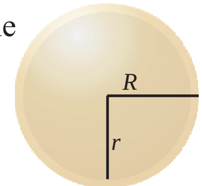


Fig. 8.39

(iii) Volume of a solid hemisphere

Volume of the solid hemisphere $= \frac{1}{2} \times$ volume of the sphere

$$= \frac{1}{2} \times \frac{4}{3}\pi r^3$$

$$= \frac{2}{3}\pi r^3 \text{ cu.units.}$$

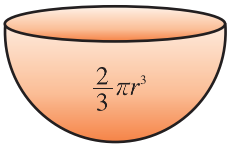


Fig. 8.40

(iv) Volume of a hollow hemisphere (Volume of the material used)

$$\left.\begin{matrix}\text{Volume of a hollow}\\\text{hemisphere}\end{matrix}\right\} = \left.\begin{matrix}\text{Volume of outer}\\\text{hemisphere}\end{matrix}\right\} - \left\{\begin{matrix}\text{Volume of inner}\\\text{hemisphere}\end{matrix}\right.$$

$$= \frac{2}{3} \times \pi \times R^3 - \frac{2}{3} \times \pi \times r^3$$

$$= \frac{2}{3}\pi (R^3 - r^3) \text{ cu.units.}$$

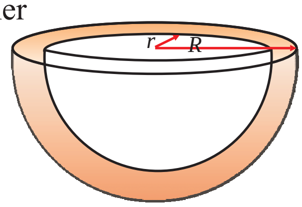


Fig. 8.41

Example 8.17

Find the volume of a sphere-shaped metallic shot-put having diameter of 8.4 cm. (Take $\pi = \frac{22}{7}$)

Solution Let r be radius of the metallic shot-put.

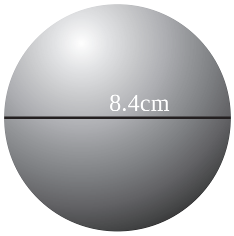


Fig. 8.42

Now, $2r = 8.4$ cm $\implies r = 4.2$ cm

Volume of the shot-put, $V = \frac{4}{3}\pi r^3$

$$= \frac{4}{3} \times \frac{22}{7} \times \frac{42}{10} \times \frac{42}{10} \times \frac{42}{10}$$

Thus, the volume of the shot-put $= 310.464$ cu.cm.

Example 8.18

A cone, a hemisphere and cylinder have equal bases. If the heights of the cone and a cylinder are equal and are same as the common radius, then find the ratio of their respective volumes.

Solution Let r be the common radius of the cone, hemisphere and cylinder.

Let h be the common height of the cone and cylinder.

Given that $r = h$

Let V_1, V_2 and V_3 be the volumes of the cone, hemisphere and cylinder respectively.

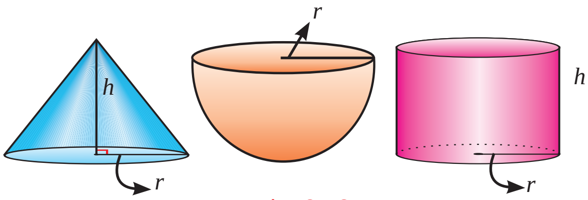


Fig. 8.43

Now, $V_1 : V_2 : V_3 = \frac{1}{3}\pi r^2 h : \frac{2}{3}\pi r^3 : \pi r^2 h$

$\implies \quad = \frac{1}{3}\pi r^3 : \frac{2}{3}\pi r^3 : \pi r^3 \qquad$ (here, $r = h$)

$\implies V_1 : V_2 : V_3 = \frac{1}{3} : \frac{2}{3} : 1$

Hence, the required ratio is $1 : 2 : 3$.

Example 8.19

If the volume of a solid sphere is $7241\frac{1}{7}$ cu.cm, then find its radius. (Take $\pi = \frac{22}{7}$)

Solution Let r and V be the radius and volume of the solid sphere respectively.

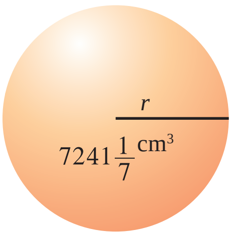


Fig. 8.44

Given that $V = 7241\frac{1}{7}$ cu.cm

$\implies \frac{4}{3}\pi r^3 = \frac{50688}{7}$

$\implies \frac{4}{3} \times \frac{22}{7} \times r^3 = \frac{50688}{7}$

$$r^3 = \frac{50688}{7} \times \frac{3 \times 7}{4 \times 22}$$
$$= 1728 = 4^3 \times 3^3$$

Thus, the radius of the sphere, $r = 12$ cm.

Example 8.20

Volume of a hollow sphere is $\frac{11352}{7}$ cm^3. If the outer radius is 8 cm, find the inner radius of the sphere. (Take $\pi = \frac{22}{7}$)

Solution Let R and r be the outer and inner radii of the hollow sphere respectively.

Let V be the volume of the hollow sphere.

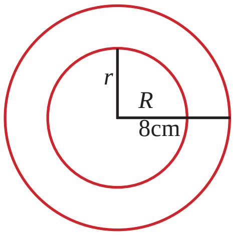


Fig. 8.45

Now, given that $V = \frac{11352}{7}$ cm^3

$$\Longrightarrow \quad \frac{4}{3}\pi(R^3 - r^3) = \frac{11352}{7}$$

$$\Longrightarrow \quad \frac{4}{3} \times \frac{22}{7}(8^3 - r^3) = \frac{11352}{7}$$

$$512 - r^3 = 387 \Longrightarrow r^3 = 125 = 5^3$$

Hence, the inner radius, $r = 5$ cm.

Exercise 8.2

1. Find the volume of a solid cylinder whose radius is 14 cm and height 30 cm.
2. A patient in a hospital is given soup daily in a cylindrical bowl of diameter 7 cm. If the bowl is filled with soup to a height of 4 cm, then find the quantity of soup to be prepared daily in the hospital to serve 250 patients?
3. The sum of the base radius and the height of a solid right circular solid cylinder is 37 cm. If the total surface area of the cylinder is 1628 sq.cm, then find the volume of the cylinder.
4. Volume of a solid cylinder is 62.37 cu.cm. Find the radius if its height is 4.5 cm.
5. The radii of two right circular cylinders are in the ratio 2 : 3. Find the ratio of their volumes if their heights are in the ratio 5 : 3.
6. The radius and height of a cylinder are in the ratio 5 : 7. If its volume is 4400 cu.cm, find the radius of the cylinder.
7. A rectangular sheet of metal foil with dimension 66 cm $\times$ 12 cm is rolled to form a cylinder of height 12 cm. Find the volume of the cylinder.
8. A lead pencil is in the shape of right circular cylinder. The pencil is 28 cm long and its radius is 3 mm. If the lead is of radius 1 mm, then find the volume of the wood used in the pencil.

9. Radius and slant height of a cone are 20 cm and 29 cm respectively. Find its volume.
10. The circumference of the base of a 12 m high wooden solid cone is 44 m. Find the volume.
11. A vessel is in the form of a frustum of a cone. Its radius at one end and the height are 8 cm and 14 cm respectively. If its volume is $\frac{5676}{3}$ cm^3, then find the radius at the other end.
12. The perimeter of the ends of a frustum of a cone are 44 cm and 8.4π cm. If the depth is 14 cm., then find its volume.
13. A right angled $\triangle ABC$ with sides 5 cm, 12 cm and 13 cm is revolved about the fixed side of 12 cm. Find the volume of the solid generated.
14. The radius and height of a right circular cone are in the ratio 2 : 3. Find the slant height if its volume is 100.48 cu.cm. (Take $\pi = 3.14$)
15. The volume of a cone with circular base is 216π cu.cm. If the base radius is 9 cm, then find the height of the cone.
16. Find the mass of 200 steel spherical ball bearings, each of which has radius 0.7 cm, given that the density of steel is 7.95 g/cm^3. (Mass = Volume $\times$ Density)
17. The outer and the inner radii of a hollow sphere are 12 cm and 10 cm. Find its volume.
18. The volume of a solid hemisphere is 1152π cu.cm. Find its curved surface area.
19. Find the volume of the largest right circular cone that can be cut out of a cube whose edge is 14 cm.
20. The radius of a spherical balloon increases from 7 cm to 14 cm as air is being pumped into it. Find the ratio of volumes of the balloon in the two cases.

8.4 Combination of Solids

In our daily life we observe many objects like toys, vehicles, vessels, tools, etc., which are combination of two or more solids.

How can we find the surface areas and volumes of combination of solids?

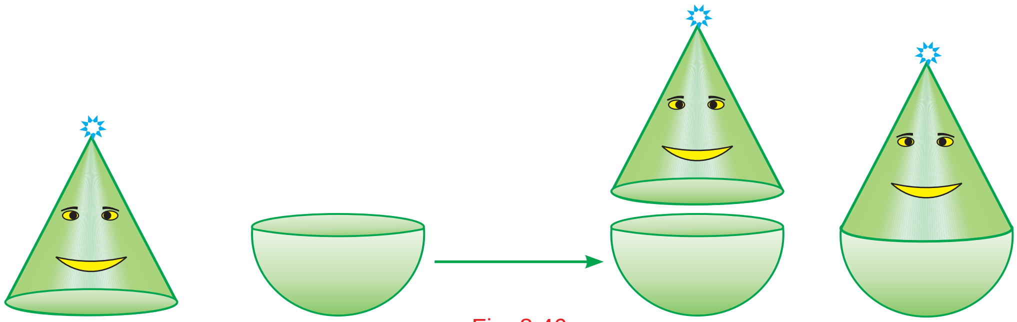


Fig. 8.46

The total surface area of the combination of solids need not be the sum of the surface areas of the solids which are combined together. However, in the above figure, the total surface area of the combined solid is equal to the sum of the curved surface area of the hemisphere and curved surface area of the cone. But the volume of the combined solid is equal to the sum of the volumes of the solids which are combined together. Thus, from the figure we have,

$$\text{The total surface area of the solid} = \left.\begin{matrix}\text{Curved surface area}\\ \text{of the hemisphere}\end{matrix}\right\} + \left\{\begin{matrix}\text{Curved surface area}\\ \text{of the cone}\end{matrix}\right.$$

The total volume of the solid = Volume of the hemisphere + Volume of the cone.

Example 8.21

A solid wooden toy is in the form of a cone surmounted on a hemisphere. If the radii of the hemisphere and the base of the cone are 3.5 cm each and the total height of the toy is 17.5 cm, then find the volume of wood used in the toy. (Take $\pi = \frac{22}{7}$)

Solution Hemispherical portion :

Radius, $r = 3.5$ cm

Conical portion :

Radius, $r = 3.5$ cm

Height, $h = 17.5 - 3.5 = 14$ cm

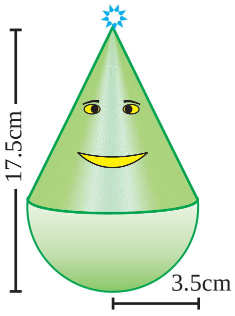


Fig. 8.47

Volume of the wood = Volume of the hemisphere + Volume of the cone

$$= \frac{2}{3}\pi r^3 + \frac{1}{3}\pi r^2 h$$

$$= \frac{\pi r^2}{3}(2r + h)$$

$$= \frac{22}{7} \times \frac{3.5 \times 3.5}{3} \times (2 \times 3.5 + 14) = 269.5$$

Hence, the volume of the wood used in the toy = 269.5 cu.cm.

Example 8.22

A cup is in the form of a hemisphere surmounted by a cylinder. The height of the cylindrical portion is 8 cm and the total height of the cup is 11.5 cm. Find the total surface area of the cup. (Take $\pi = \frac{22}{7}$)

Solution Hemispherical portion

Radius, r = Total height – 8

$\Longrightarrow r = 11.5 - 8 = 3.5$ cm

Cylindrical portion

Height, $h = 8$ cm.

Thus, radius $r = 3.5\text{ cm} = \frac{7}{2}$ cm

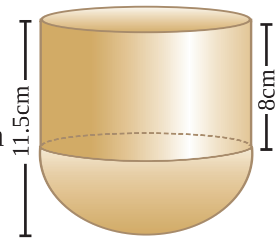


Fig. 8.48

$$\text{Total surface area of the cup} = \left\{\begin{matrix}\text{CSA of the hemispherical portion}\\ +\ \text{CSA of the cylindrical portion}\end{matrix}\right.$$

$$= 2\pi r^2 + 2\pi r h = 2\pi r(r + h)$$

$$= 2 \times \frac{22}{7} \times \frac{7}{2}\left(\frac{7}{2} + 8\right)$$

$\therefore$ Total surface area of the cup = 253 sq.cm.

Example 8.23

A circus tent is to be erected in the form of a cone surmounted on a cylinder. The total height of the tent is 49 m. Diameter of the base is 42 m and height of the cylinder is 21 m. Find the cost of canvas needed to make the tent, if the cost of canvas is ₹12.50/m^2. (Take $\pi = \frac{22}{7}$)

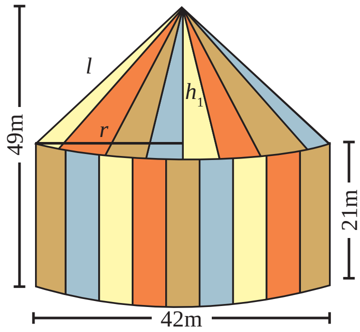


Fig. 8.49

Solution

Cylindrical Part	Conical Part
Diameter, $2r = 42$ m	Radius, $r = 21$ m
Radius, $r = 21$ m	Height, $h_1 = 49 - 21 = 28$ m
Height, $h = 21$ m	Slant height, $l = \sqrt{h_1^2 + r^2}$
	$= \sqrt{28^2 + 21^2}$
	$= 7\sqrt{4^2 + 3^2} = 35$ m

Total area of the canvas needed = CSA of the cylindrical part + CSA of the conical part

$$= 2\pi rh + \pi rl = \pi r(2h + l)$$

$$= \frac{22}{7} \times 21(2 \times 21 + 35) = 5082$$

Therefore, area of the canvas = 5082 m^2

Now, the cost of the canvas per sq.m = ₹12.50

Thus, the total cost of the canvas = 5082 × 12.5 = ₹63525.

Example 8.24

A hollow sphere of external and internal diameters of 8 cm and 4 cm respectively is melted and made into another solid in the shape of a right circular cone of base diameter of 8 cm. Find the height of the cone.

Solution Let R and r be the external and internal radii of the hollow sphere.

Let h and r_1 be the height and the radius of the cone to be made.

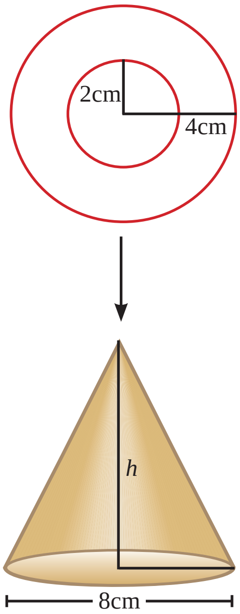


Fig. 8.50

Hollow Sphere External	Hollow Sphere Internal	Cone
$2R = 8$ cm	$2r = 4$ cm	$2r_1 = 8$
$\Longrightarrow R = 4$ cm	$\Longrightarrow r = 2$ cm	$\Longrightarrow r_1 = 4$

When the hollow sphere is melted and made into a solid cone, we have

Volume of the cone = Volume of the hollow sphere

$$\Longrightarrow \quad \frac{1}{3}\pi r_1^2 h = \frac{4}{3}\pi[R^3 - r^3]$$

$$\Longrightarrow \quad \frac{1}{3}\times\pi\times 4^2\times h = \frac{4}{3}\times\pi\times(4^3-2^3)$$

$$\Longrightarrow \quad h = \frac{64-8}{4} = 14$$

Hence, the height of the cone $h = 14$ cm.

Example 8.25

Spherical shaped marbles of diameter 1.4 cm each, are dropped into a cylindrical beaker of diameter 7 cm containing some water. Find the number of marbles that should be dropped into the beaker so that the water level rises by 5.6 cm.

Solution Let n be the number of marbles needed. Let r_1 and r_2 be the radii of the marbles and cylindrical beaker respectively.

Marbles	Cylindrical Beaker
Diameter, $2r_1 = 1.4$ cm	Diameter, $2r_2 = 7$ cm
Radius $r_1 = 0.7$ cm	Radius, $r_2 = \frac{7}{2}$ cm

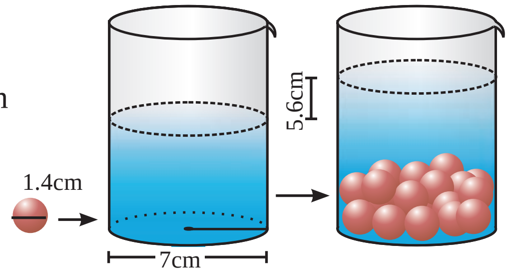


Fig. 8.51

Let h be the height of the water level raised.

Then, $h = 5.6$ cm

After the marbles are dropped into the beaker,

Volume of water raised = Volume of n marbles

$$\Longrightarrow \quad \pi r_2^2 h = n\times\frac{4}{3}\pi r_1^3$$

Thus, $$n = \frac{3r_2^2 h}{4r_1^3}$$

$$n = \frac{3\times\frac{7}{2}\times\frac{7}{2}\times 5.6}{4\times\frac{7}{10}\times\frac{7}{10}\times\frac{7}{10}} = 150.$$

$\therefore$ The number of marbles needed is 150.

Example 8.26

Water is flowing at the rate of 15 km / hr through a cylindrical pipe of diameter 14 cm into a rectangular tank which is 50 m long and 44 m wide. In how many hours will the water level in the tank raise by 21 cm? (Take $\pi = \frac{22}{7}$)

Solution Speed of water = 15 km / hr

= 15000 m / hr

Diameter of the pipe, $2r = 14$ cm

Thus, $r = \frac{7}{100}$ m.

Let h be the water level to be raised.

Thus, $h = 21 \text{ cm} = \frac{21}{100}$ m

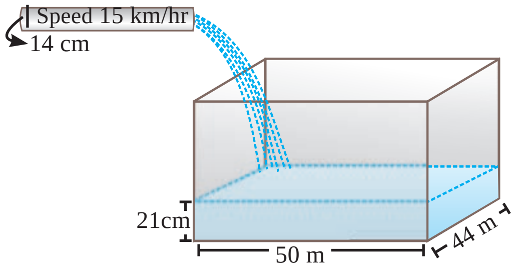


Fig. 8.52

Now, the volume of water discharged

$$= \text{Cross section area of the pipe} \times \text{Time} \times \text{Speed}$$

Volume of water discharged in one hour

$$= \pi r^2 \times 1 \times 15000$$
$$= \frac{22}{7} \times \frac{7}{100} \times \frac{7}{100} \times 15000 \text{ cu.m}$$

Volume of required quantity of water in the tank is,

$$lbh = 50 \times 44 \times \frac{21}{100}$$

Assume that T hours are needed to get the required quantity of water.

$$\therefore \quad \left.\begin{matrix}\text{Volume of water discharged} \\ \text{in T hours}\end{matrix}\right\} = \text{Required quantity of water in the tank}$$

$$\Longrightarrow \quad \frac{22}{7} \times \left(\frac{7}{100}\right)^2 \times T \times 15000 = 50 \times 44 \times \frac{21}{100}$$

Thus, $\qquad T = 2$ hours.

Hence, it will take 2 hours to raise the required water level.

Example 8.27

A cuboid shaped slab of iron whose dimensions are $55\,\text{cm} \times 40\,\text{cm} \times 15\,\text{cm}$ is melted and recast into a pipe. The outer diameter and thickness of the pipe are 8 cm and 1 cm respectively. Find the length of the pipe. $\left(\text{Take } \pi = \frac{22}{7}\right)$

Solution Let h_1 be the length of the pipe.

Let R and r be the outer and inner radii of the pipe respectively.

Iron slab: Let $lbh = 55 \times 40 \times 15$.

Iron pipe:

Outer diameter, $\quad 2R = 8\,\text{cm}$

$\therefore$ Outer radius, $\quad R = 4\,\text{cm}$

Thickness, $\quad w = 1\,\text{cm}$

$\therefore$ Inner radius, $\quad r = R - w = 4 - 1 = 3\,\text{cm}$

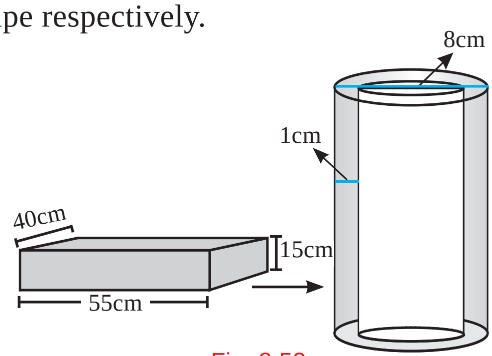


Fig. 8.53

Now, the volume of the iron pipe = Volume of iron slab

$$\Longrightarrow \quad \pi h_1 (R + r)(R - r) = lbh$$

That is, $\qquad \frac{22}{7} \times h_1 (4 + 3)(4 - 3) = 55 \times 40 \times 15$

Thus, the length of the pipe, $\quad h_1 = 1500\,\text{cm} = 15\,\text{m}$.

Exercise 8.3

1. A play-top is in the form of a hemisphere surmounted on a cone. The diameter of the hemisphere is 3.6 cm. The total height of the play-top is 4.2cm. Find its total surface area.
2. A solid is in the shape of a cylinder surmounted on a hemisphere. If the diameter and the total height of the solid are 21 cm, 25.5 cm respectively, then find its volume.
3. A capsule is in the shape of a cylinder with two hemispheres stuck to each of its ends. If the length of the entire capsule is 14 mm and the diameter of the capsule is 5 mm, find its surface area.
4. A tent is in the shape of a right circular cylinder surmounted by a cone. The total height and the diameter of the base are 13.5 m and 28 m. If the height of the cylindrical portion is 3 m, find the total surface area of the tent.
5. Using clay, a student made a right circular cone of height 48 cm and base radius 12 cm. Another student reshapes it in the form of a sphere. Find the radius of the sphere.
6. The radius of a solid sphere is 24 cm. It is melted and drawn into a long wire of uniform cross section. Find the length of the wire if its radius is 1.2 mm.
7. A right circular conical vessel whose internal radius is 5 cm and height is 24 cm is full of water. The water is emptied into an empty cylindrical vessel with internal radius 10 cm. Find the height of the water level in the cylindrical vessel.
8. A solid sphere of diameter 6 cm is dropped into a right circular cylindrical vessel with diameter 12 cm, which is partly filled with water. If the sphere is completely submerged in water, how much does the water level in the cylindrical vessel increase?.
9. Through a cylindrical pipe of internal radius 7 cm, water flows out at the rate of 5 cm/sec. Calculate the volume of water (in litres) discharged through the pipe in half an hour.
10. Water in a cylindrical tank of diameter 4 m and height 10 m is released through a cylindrical pipe of diameter 10 cm at the rate of 2.5 Km/hr. How much time will it take to empty the half of the tank? Assume that the tank is full of water to begin with.
11. A spherical solid material of radius 18 cm is melted and recast into three small solid spherical spheres of different sizes. If the radii of two spheres are 2cm and 12 cm, find the radius of the third sphere.
12. A hollow cylindrical pipe is of length 40 cm. Its internal and external radii are 4 cm and 12 cm respectively. It is melted and cast into a solid cylinder of length 20 cm. Find the radius of the new solid.
13. An iron right circular cone of diameter 8 cm and height 12 cm is melted and recast into spherical lead shots each of radius 4 mm. How many lead shots can be made?.

14. A right circular cylinder having diameter 12 cm and height 15 cm is full of ice cream. The ice cream is to be filled in cones of height 12 cm and diameter 6 cm, having a hemispherical shape on top. Find the number of such cones which can be filled with the ice cream available.

15. A container with a rectangular base of length 4.4 m and breadth 2 m is used to collect rain water. The height of the water level in the container is 4 cm and the water is transferred into a cylindrical vessel with radius 40 cm. What will be the height of the water level in the cylinder?

16. A cylindrical bucket of height 32 cm and radius 18 cm is filled with sand. The bucket is emptied on the ground and a conical heap of sand is formed. If the height of the conical heap is 24 cm, find the radius and slant height of the heap.

17. A cylindrical shaped well of depth 20 m and diameter 14 m is dug. The dug out soil is evenly spread to form a cuboid-platform with base dimension 20 m $\times$ 14 m. Find the height of the platform.

Exercise 8.4

Choose the correct answer

1. The curved surface area of a right circular cylinder of radius 1 cm and height 1 cm is equal to

   (A) π cm^2 (B) 2π cm^2 (C) 3π cm^3 (D) 2 cm^2

2. The total surface area of a solid right circular cylinder whose radius is half of its height h is equal to

   (A) $\frac{3}{2}\pi h$ sq. units (B) $\frac{2}{3}\pi h^2$ sq. units (C) $\frac{3}{2}\pi h^2$ sq.units (D) $\frac{2}{3}\pi h$ sq.units

3. Base area of a right circular cylinder is 80 cm^2 . If its height is 5 cm, then the volume is equal to

   (A) 400 cm^3 (B) 16 cm^3 (C) 200 cm^3 (D) $\frac{400}{3}$ cm^3

4. If the total surface area a solid right circular cylinder is 200π cm^2 and its radius is 5 cm, then the sum of its height and radius is

   (A) 20 cm (B) 25 cm (C) 30 cm (D) 15 cm

5. The curved surface area of a right circular cylinder whose radius is a units and height is b units, is equal to

   (A) $\pi a^2 b$ sq.cm (B)$2\pi ab$ sq.cm (C) 2π sq.cm (D) 2 sq.cm

6. Radius and height of a right circular cone and that of a right circular cylinder are respectively, equal. If the volume of the cylinder is 120 cm^3, then the volume of the cone is equal to

   (A) 1200 cm^3 B) 360 cm^3 (C) 40 cm^3 (D) 90 cm^3

7. If the diameter and height of a right circular cone are 12 cm and 8 cm respectively, then the slant height is

(A) 10 cm (B) 20 cm (C) 30 cm (D) 96 cm

8. If the circumference at the base of a right circular cone and the slant height are 120π cm and 10 cm respectively, then the curved surface area of the cone is equal to

(A) 1200π cm^2 (B) 600π cm^2 (C) 300π cm^2 (D) 600 cm^2

9. If the volume and the base area of a right circular cone are 48π cm^3 and 12π cm^2 respectively, then the height of the cone is equal to

(A) 6 cm (B) 8 cm (C) 10 cm (D) 12 cm

10. If the height and the base area of a right circular cone are 5 cm and 48 sq.cm respectively, then the volume of the cone is equal to

(A) 240 cm^3 (B) 120 cm^3 (C) 80 cm^3 (D) 480 cm^3

11. The ratios of the respective heights and the respective radii of two cylinders are 1:2 and 2:1 respectively. Then their respective volumes are in the ratio

(A) 4 : 1 (B) 1 : 4 (C) 2 : 1 (D) 1 : 2

12. If the radius of a sphere is 2 cm , then the curved surface area of the sphere is equal to

(A) 8π cm^2 (B) 16 cm^2 (C) 12π cm^2 (D) 16π cm^2 .

13. The total surface area of a solid hemisphere of diameter 2 cm is equal to

(A) 12 cm^2 (B) 12π cm^2 (C) 4π cm^2 (D) 3π cm^2.

14. If the volume of a sphere is $\frac{9}{16}\pi$ cu.cm, then its radius is

(A) $\frac{4}{3}$ cm (B) $\frac{3}{4}$ cm (C) $\frac{3}{2}$ cm (D) $\frac{2}{3}$ cm.

15. The surface areas of two spheres are in the ratio of $9:25$. Then their volumes are in the ratio

(A) 81 : 625 (B) 729 : 15625 (C) 27 : 75 (D) 27 : 125.

16. The total surface area of a solid hemisphere whose radius is a units, is equal to

(A) $2\pi a^2$ sq.units (B) $3\pi a^2$ sq.units (C) $3\pi a$ sq.units (D) $3a^2$ sq.units.

17. If the surface area of a sphere is 100π cm^2, then its radius is equal to

(A) 25 cm (B) 100 cm (C) 5 cm (D) 10 cm .

18. If the surface area of a sphere is 36π cm^2, then the volume of the sphere is equal to

(A) 12π cm^3 (B) 36π cm^3 (C) 72π cm^3 (D) 108π cm^3.

19. If the total surface area of a solid hemisphere is 12π cm^2 then its curved surface area is equal to

(A) 6π cm^2 (B) 24π cm^2 (C) 36π cm^2 (D) 8π cm^2.

20. If the radius of a sphere is half of the radius of another sphere, then their respective volumes are in the ratio

(A) 1 : 8 (B) 2: 1 (C) 1 : 2 (D) 8 : 1

21. Curved surface area of solid sphere is 24 cm^2. If the sphere is divided into two hemispheres, then the total surface area of one of the hemispheres is

(A) 12 cm^2 (B) 8 cm^2 (C) 16 cm^2 (D) 18 cm^2

22. Two right circular cones have equal radii. If their slant heights are in the ratio 4 : 3, then their respective curved surface areas are in the ratio

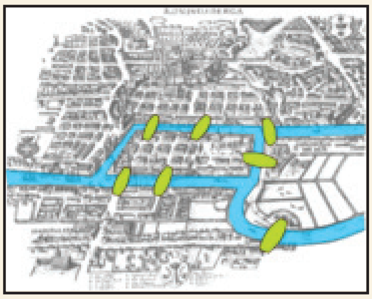
(A) 16 : 9 (B) 2 : 3 (C) 4 : 3 (D) 3 : 4

Do you know?

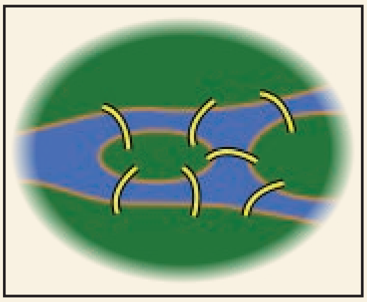
The Seven Bridges of Königsberg is a notable historical problem in mathematics. The city of Königsberg in Prussia (now Kaliningrad, Russia) was set on both sides of the Pregel River, and included two large islands which were connected to each other and the mainland by seven bridges.(See Figure)

The problem was to find a route through the city that would cross each bridge once and only once. The islands could not be reached by any route other than the bridges, and every bridge must have been crossed completely every time (one could not walk half way onto the bridge and then turn around and later cross the other half from the other side).

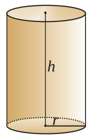
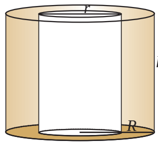
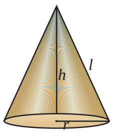
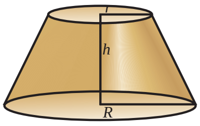
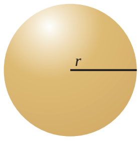
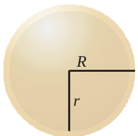
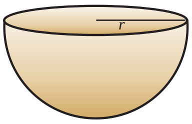
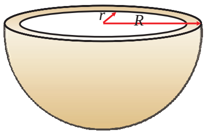
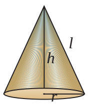
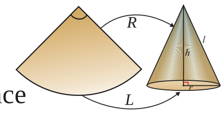
Leonhard Euler in 1735 proved that the problem has no solution. Its negative resolution by Euler laid the foundations of graph theory and presaged the idea of topology.



$\Longleftrightarrow$



Points to Remember

Sl. No	Name	Figure	Lateral or Curved Surface Area (sq.units)	Total Surface Area (sq.units)	Volume (cu.units)
1	**Solid right circular cylinder**		$2\pi rh$	$2\pi r(h+r)$	$\pi r^2 h$
2	**Right circular hollow cylinder**		$2\pi h(R+r)$	$2\pi(R+r)(R-r+h)$	Volume of the material used $\pi R^2 h - \pi r^2 h$ $= \pi h(R^2 - r^2)$ $= \pi h(R+r)(R-r)$
3	**Solid right circular cone**		πrl	$\pi r(l+r)$	$\frac{1}{3}\pi r^2 h$
4	**Frustum**		- - - - - - -	- - - - - - - - - - - - - - -	$\frac{1}{3}\pi h(R^2 + r^2 + Rr)$
5	**Sphere**		$4\pi r^2$	- - -	$\frac{4}{3}\pi r^3$
6	**Hollow sphere**		- - -	- - -	Volume of the material used $\frac{4}{3}\pi(R^3 - r^3)$
7	**Solid Hemisphere**		$2\pi r^2$	$3\pi r^2$	$\frac{2}{3}\pi r^3$
8	**Hollow Hemisphere**		$2\pi(R^2 + r^2)$	$2\pi(R^2 + r^2) + \pi(R^2 - r^2)$ $= \pi(3R^2 + r^2)$	Volume of the material used $\frac{2}{3}\pi(R^3 - r^3)$
9	**A sector of a circle converted into a Cone**  CSA of a cone = Area of the sector $\pi rl = \frac{\theta}{360} \times \pi r^2$ Length of the sector = Base circumference of the cone	$l = \sqrt{h^2 + r^2}$ $h = \sqrt{l^2 - r^2}$ $r = \sqrt{l^2 - h^2}$ 		10. Volume of water flows out through a pipe = {Cross section area $\times$ Speed $\times$ Time } 11. No. of new solids obtained by recasting $= \dfrac{\text{Volume of the solid which is melted}}{\text{volume of one solid which is made}}$		
12	Conversions	1 m^3 = 1000 litres , 1 d.m^3 = 1 litre , 1000 cm^3 = 1 litre , 1000 litres = 1 *kl*				

9

PRACTICAL GEOMETRY

Give me a place to stand, and I shall move the earth
-Archimedes

- Introduction
- Tangents
- Triangles
- Cyclic Quadrilaterals

Brahmagupta

(598-668 AD)

India

(Great Scientist of Ancient India)

Brahmagupta wrote the book "Brahmasphuta Siddhanta". His most famous result in geometry is a formula for cyclic quadrilateral :

Given the lengths p, q, r and s of the sides of any cyclic quadrilateral, he gave an approximate and an exact formula for the area.

Approximate area is

$$\left(\frac{p+r}{2}\right)\left(\frac{q+s}{2}\right).$$

Exact area is

$$\sqrt{(t-p)(t-q)(t-r)(t-s)},$$

where $2t = p+q+r+s$.

9.1 Introduction

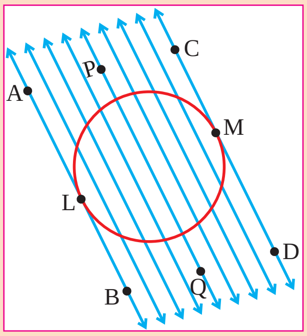
Geometry originated in Egypt as early as 3000 B.C., was used for the measurement of land. Early geometry was a collection of empirically discovered principles concerning lengths, angles, areas, and volumes which were developed to meet some practical needs in surveying, construction, astronomy and various other crafts.

Recently there have been several new efforts to reform curricula to make geometry less worthy than its counterparts such as algebra, analysis, etc. But many mathematicians strongly disagree with this reform. In fact, geometry helps in understanding many mathematical ideas in other parts of mathematics. In this chapter, we shall learn how to draw tangents to circles, triangles and cyclic quadrilaterals with the help of given actual measurements.

In class IX, we have studied about various terms related to circle such as chord, segment, sector, etc. Let us recall some of the terms like secant, tangent to a circle through the following activities.

Activity

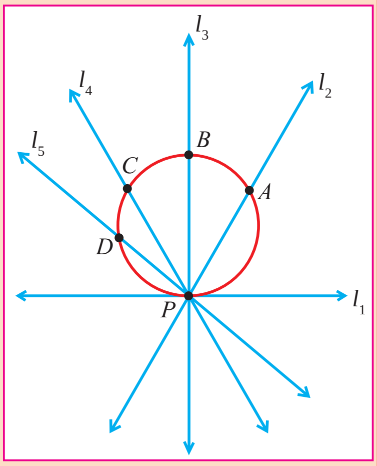
Take a paper and draw a circle of any radius. Draw a secant PQ to the circle. Now draw as many secants as possible parallel to PQ on both sides of PQ. Note that the points of contact of the secants are coming closer and closer on



either side. You can also note that at one stage, the two points will coincide on both sides. Among the secants parallel to PQ, the straight lines AB and CD, just touch the circle exactly at one point on the circle, say at L and M respectively. These lines AB, CD are called **tangents** to the circle at L, M respectively. We observe that AB is parallel to CD.

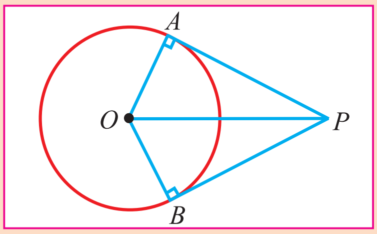
Activity

Let us draw a circle and take a point P on the circle. Draw many lines through the point P as shown in the figure. The straight lines which are passing through P, have two contact points on the circle. The straight lines l_2, l_3, l_4 and l_5 meet the circle at A, B, C and D respectively. So these lines l_2, l_3, l_4, l_5 are the secants to the circle. But the line l_1 touches the circle exactly at one point P. Now the line l_1 is called the **tangent** to the circle at P.



We know that in a circle, the radius drawn at the point of contact is perpendicular to the tangent at that point.

Let AP be a tangent at A drawn from an external point P to a circle



In a right angled ΔOPA , $OA \perp AP$

$$OP^2 = OA^2 + AP^2 \quad \text{[By Pythagoras theorem]}$$

$$AP = \sqrt{OP^2 - OA^2}.$$

9.2 Construction of tangents to a circle

Now let us learn how to draw a tangent to a circle

(i) using centre

(ii) using tangent-chord theorem .

9.2.1 Construction of a tangent to a circle (using the centre)

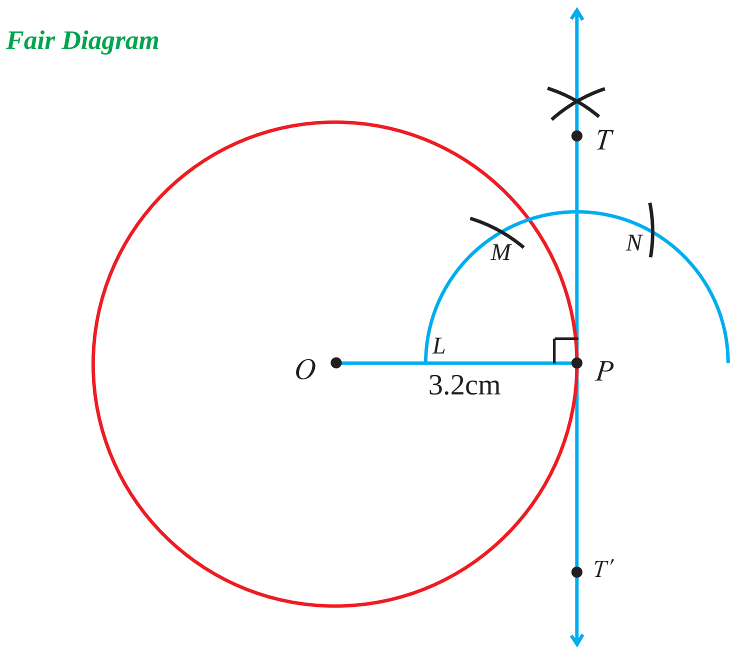
Result

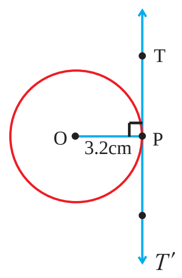
In a circle, the radius drawn at the point of contact is perpendicular to the tangent at that point.

Example 9.1

Draw a circle of radius 3.2cm. Take a point P on this circle and draw a tangent at P. (using the centre)

Given: Radius of the circle = 3.2 cm.





Construction

(i) With O as the centre draw a circle of radius 3.2 cm.

(ii) Take a point P on the circle and join OP.

(iii) Draw an arc of a circle with centre at P cutting OP at L.

(iv) Mark M and N on the arc such that $\overset{\frown}{LM} = \overset{\frown}{MN} = LP$.

(v) Draw the bisector PT of the angle $\angle MPN$.

(vi) Produce TP to T' to get the required tangent $T'PT$.

Remarks

One can draw the perpendicular line PT to the straight line OP through the point P on the circle. Now, PT is the tangent to the circle at the point P.

9.2.2 Construction of a tangent to a circle using the tangent-chord theorem

Result **The tangent-chord theorem**

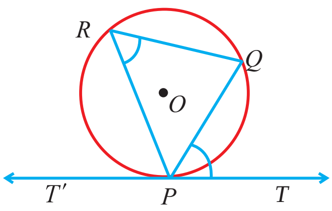
The angle between a chord of a circle and the tangent at one end of the chord is equal to the angle subtended by the chord on the alternate segment of the circle.

Example 9.2

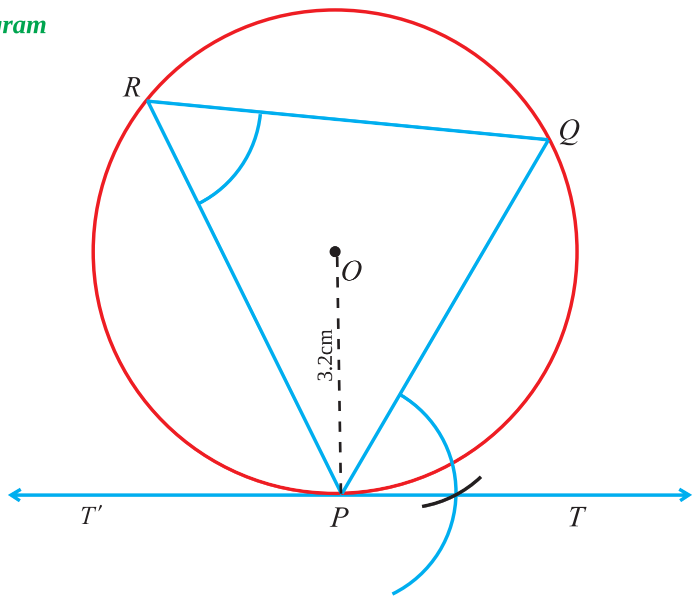
Draw a circle of radius 3.2 cm At a point P on it, draw a tangent to the circle using the tangent-chord theorem.

Given : The radius of the circle = 3.2 cm.

Rough Diagram



Fair Diagram



Construction

(i) With O as the centre, draw a circle of radius 3.2 cm.

(ii) Take a point P on the circle.

(iii) Through P, draw any chord PQ.

(iv) Mark a point R distinct from P and Q on the circle so that P, Q and R are in counter clockwise direction.

(v) Join PR and QR.

(vi) At P, construct $\angle QPT = \angle PRQ$.

(vii) Produce TP to T' to get the required tangent line $T'PT$.

9.2.3 Construction of pair of tangents to a circle from an external point

Results

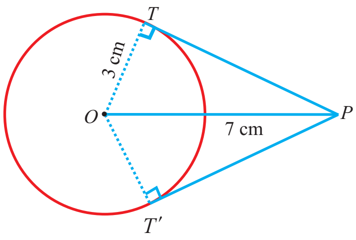
(i) Two tangents can be drawn to a circle from an external point.

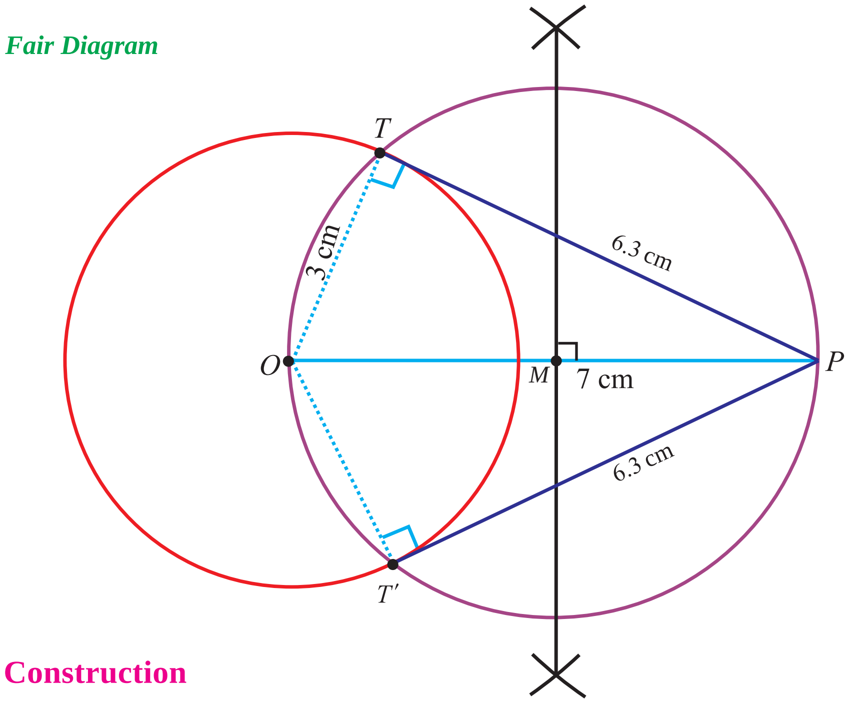
(ii) Diameters subtend 90° on the circumference of a circle.

Example 9.3

Draw a circle of radius 3cm. From an external point 7cm away from its centre, construct the pair of tangents to the circle and measure their lengths.

Given: Radius of the circle = 3 cm. OP = 7 cm.





Construction

(i) With O as the centre draw a circle of radius 3 cm.

(ii) Mark a point P at a distance of 7 cm from O and join OP.

(iii) Draw the perpendicular bisector of OP. Let it meet OP at M.

(iv) With M as centre and MO as radius, draw another circle.

(v) Let the two circles intersect at T and T'.

(vi) Join PT and PT'. They are the required tangents.

Length of the tangent, PT = 6.3 cm

Verification

In the right angled ΔOPT,

$$PT = \sqrt{OP^2 - OT^2} = \sqrt{7^2 - 3^2}$$

$$= \sqrt{49 - 9} = \sqrt{40} \qquad \therefore \quad PT = 6.3 \text{ cm (approximately)}.$$

Exercise 9.1

1. Draw a circle of radius 4.2 cm, and take any point on the circle. Draw the tangent at that point using the centre.

2. Draw a circle of radius 4.8 cm. Take a point on the circle. Draw the tangent at that point using the tangent-chord theorem.

3. Draw a circle of diameter 10 cm. From a point *P*, 13 cm away from its centre, draw the two tangents *PA* and *PB* to the circle, and measure their lengths.

4. Draw the two tangents from a point which is 10 cm away from the centre of a circle of radius 6 cm. Also, measure the lengths of the tangents.

5. Take a point which is 9 cm away from the centre of a circle of radius 3 cm, and draw the two tangents to the circle from that point.

9.3 Construction of triangles

We have already learnt how to construct triangles when sides and angles are given. In this section, let us construct a triangle when

(i) the base, vertical angle and the altitude from the vertex to the base are given.

(ii) the base, vertical angle and the median from the vertex to the base are given.

First, let us describe the way of constructing a segment of a circle on a given line segment containing a given angle.

Construction of a segment of a circle on a given line segment containing an angle θ

Construction

(i) Draw a line segment $\overline{BC}$.

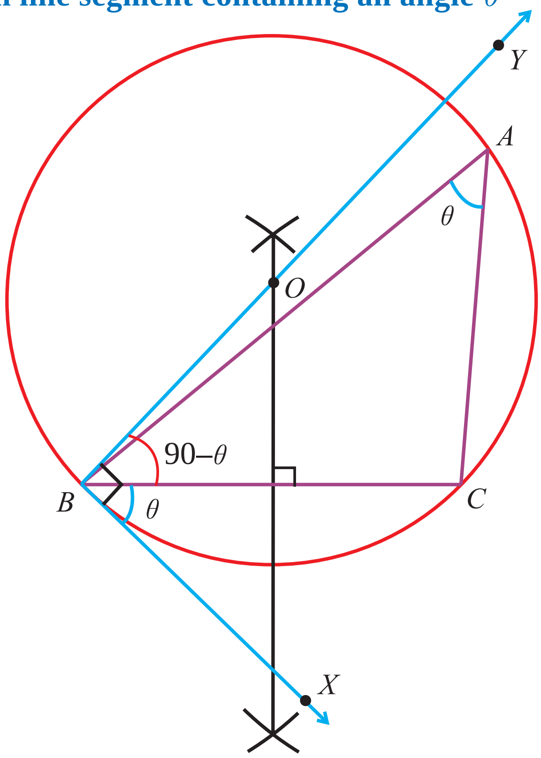
(ii) At B, make $\angle CBX = \theta$.

(iii) Draw $BY \perp BX$.

(iv) Draw the perpendicular bisector of BC which meets BY at O.

(v) With O as centre and OB as radius draw a circle.

(vi) Take any point A on the circle. By the tangent-chord theorem, the major arc BAC is the required segment of the circle containing the angle θ.



Construction of a triangle when its base and the vertical angle are given.

We shall describe the various steps involved in the construction of a triangle when its base and the vertical angle are given.

Construction

(i) Draw a line segment AB.

(ii) At A, make the given angle $\angle BAX = \theta$

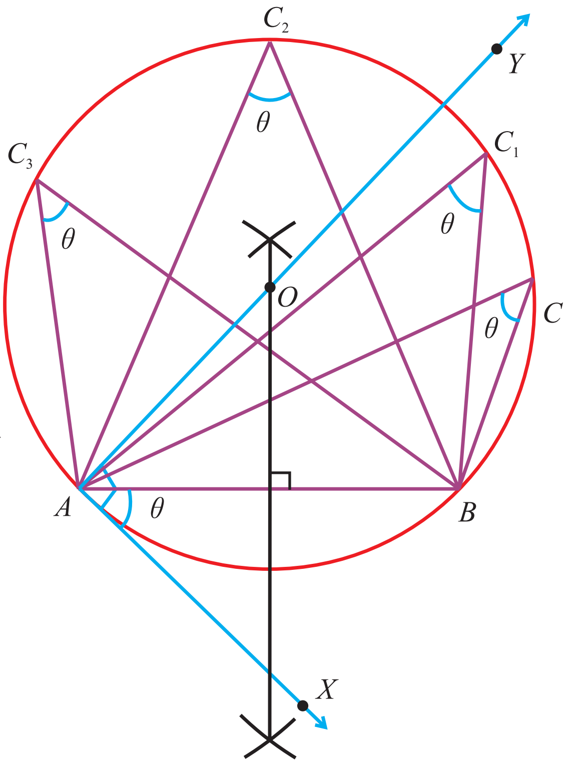
(iii) Draw $AY \perp AX$.

(iv) Draw the perpendicular bisector of AB which meets AY at O.

(v) With O as centre OA as radius, draw a circle.

(vi) Take any point C on the alternate segment of the circle and join AC and BC.

(vii) $\triangle ABC$ is the required triangle.



Now, one can justify that $\triangle ABC$ is one of the triangles, with the given base and the vertical angle.

Note that $AX \perp AY$. Thus, $\angle XAY = 90^\circ$.

Also, $OB = OA$. (the radii of the circle).

AX is the tangent to the circle at A and C is any point on the circle.

Hence, $\angle BAX = \angle ACB$. (tangent-chord theorem).

Remarks

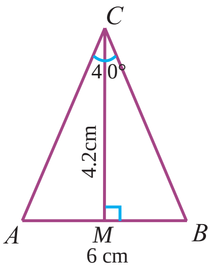
If we take $C_1, C_2, C_3, \ldots$ are points on the circle, then all the triangle ΔABC_1, ΔABC_2, ΔABC_3, $\cdots$ are with same base and the same vertical angle.

9.3.1 Construction of a triangle when its base, the vertical angle and the altitude from the vertex to the base are given.

Example 9.4

Construct a ΔABC such that $AB = 6$ cm, $\angle C = 40^{\circ}$ and the altitude from C to AB is of length 4.2 cm.

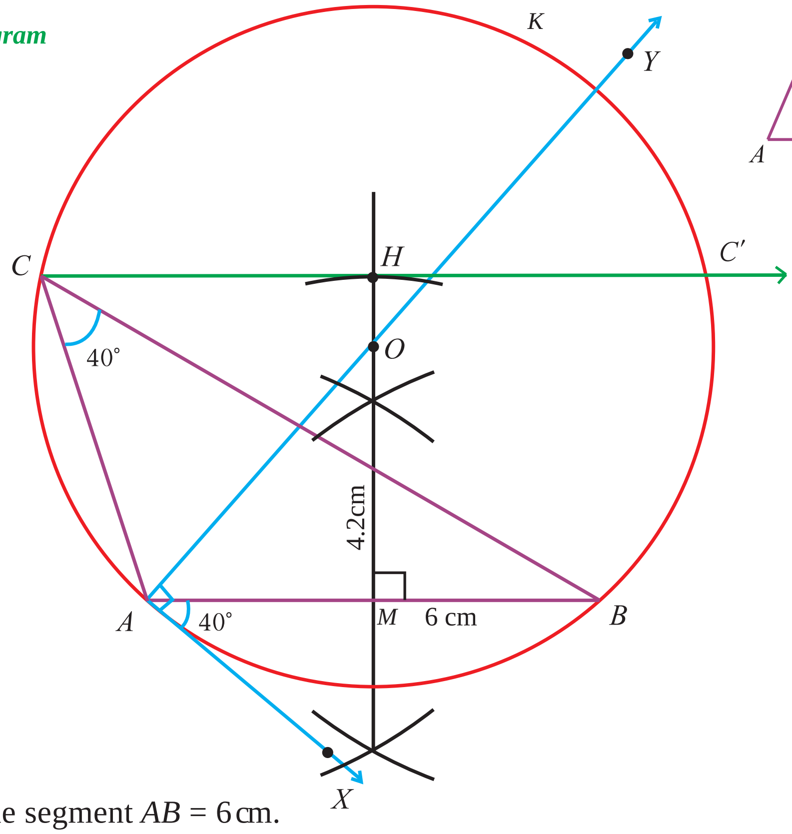
Rough Diagram



Given : In ΔABC, $AB = 6$ cm , $\angle C = 40^{\circ}$

The length of the altitude from C to AB is 4.2 cm.

Fair Diagram



Construction

(i) Draw a line segment $AB = 6$ cm.

(ii) Draw AX such that $\angle BAX = 40^{\circ}$.

(iii) Draw $AY \perp AX$.

(iv) Draw the perpendicular bisector of AB intersecting AY at O and AB at M.

(v) With O as centre and OA as radius, draw the circle .

(vi) The segment AKB contains the vertical angle 40°.

(vii) On the perpendicular bisector MO, mark a point H such that $MH = 4.2$ cm.

(viii) Draw CHC' parallel to AB meeting the circle at C and at C'.

(ix) Complete the ΔABC, which is one of the required triangles.

Remarks

$\Delta\, ABC'$ is also another required triangle.

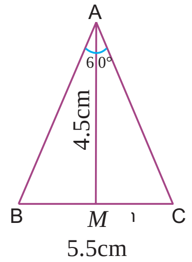
9.3.2 Construction of a triangle when its base, the vertical angle and the median from the vertex to the base are given.

Example 9.5

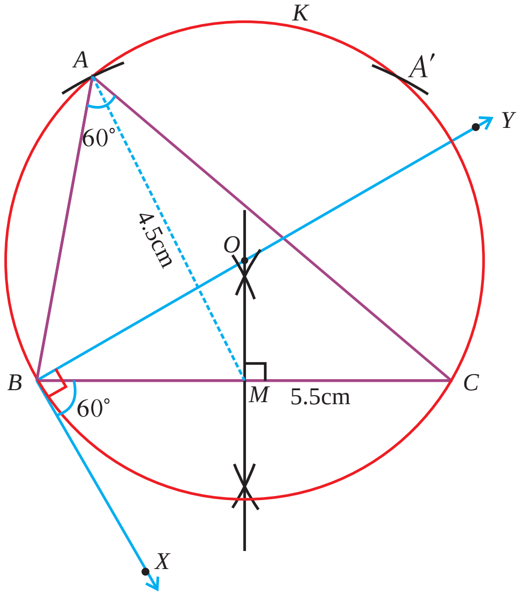
Construct a ΔABC in which $BC = 5.5$ cm., $\angle A = 60^\circ$ and the median AM from the vertex A is 4.5 cm.

Given : In ΔABC, $BC = 5.5$ cm , $\angle A = 60^\circ$, Median $AM = 4.5$ cm.

Rough Diagram



Fair Diagram



Construction

(i) Draw a line segment $BC = 5.5$ cm.

(ii) Through B draw BX such that $\angle CBX = 60^\circ$.

(iii) Draw $BY \perp BX$.

(iv) Draw the perpendicular bisector of BC intersecting BY at O and BC at M.

(v) With O as centre and OB as radius, draw the circle.

(vi) The major arc BKC of the circle, contains the vertical angle 60°.

(vii) With M as centre, draw an arc of radius 4.5 cm meeting the circle at A and A'.

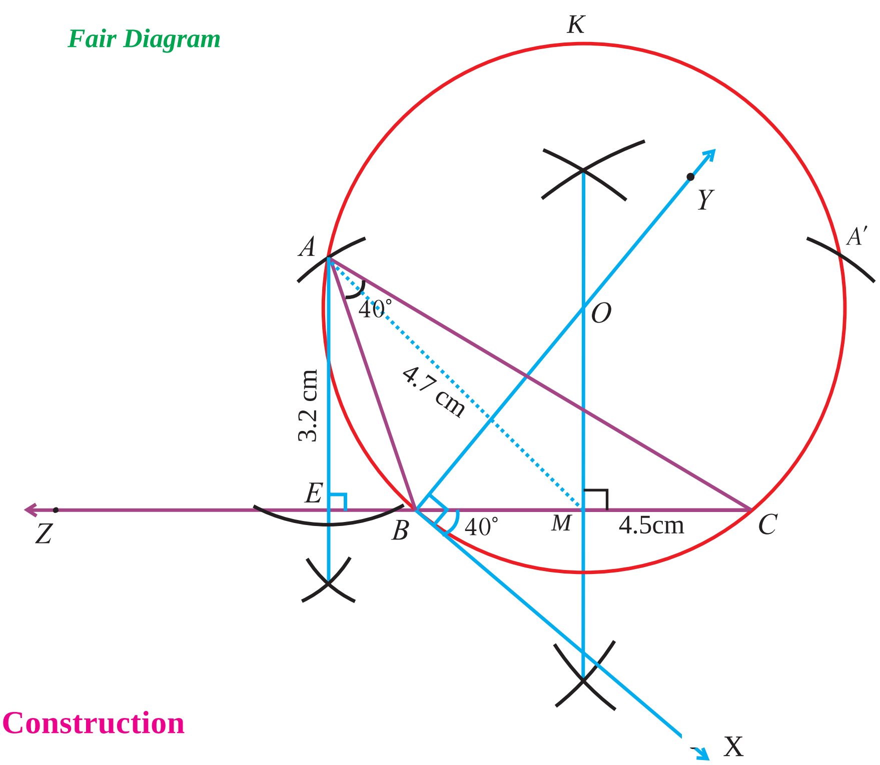
(viii) $\triangle ABC$ or $\Delta A'BC$ is the required triangle.

Example 9.6

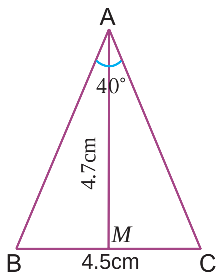
Construct a ΔABC, in which $BC = 4.5$ cm, $\angle A = 40^\circ$ and the median AM from A to BC is 4.7 cm. Find the length of the altitude from A to BC.

Given : In ΔABC, $BC = 4.5$ cm, $\angle A = 40^\circ$ and the median AM from A to BC is 4.7cm.

Fair Diagram



Rough Diagram



Construction

(i) Draw a line segment $BC = 4.5$cm.

(ii) Draw BX such that $\angle CBX = 40^\circ$.

(iii) Draw $BY \perp BX$.

(iv) Draw the perpendicular bisector of BC intersecting BY at O and BC at M.

(v) With O as centre and OB as radius, draw the circle .

(vi) The major arc BKC of the circle, contains the vertical angle 40°.

(vii) With M as centre draw an arc of radius 4.7cm meeting the circle at A and A'.

(viii) Complete ΔABC or $\Delta A'BC$, which is the required triangle.

(ix) Produce CB to CZ.

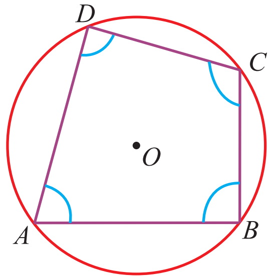
(x) Draw $AE \perp CZ$.

(xi) Length of the altitude AE is 3.2cm.

Exercise 9.2

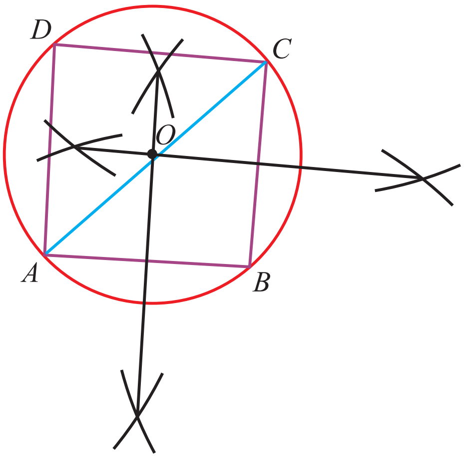
1. Construct a segment of a circle on a given line segment $AB = 5.2$ cm containing an angle 48°.
2. Construct a ΔPQR in which the base $PQ = 6$ cm, $\angle R = 60^\circ$ and the altitude from R to PQ is 4 cm.
3. Construct a ΔPQR such that $PQ = 4$ cm, $\angle R = 25^\circ$ and the altitude from R to PQ is 4.5cm.
4. Construct a ΔABC such that $BC = 5$ cm. $\angle A = 45^\circ$ and the median from A to BC is 4cm.
5. Construct a ΔABC in which the base $BC = 5$ cm, $\angle BAC = 40^\circ$ and the median from A to BC is 6 cm. Also, measure the length of the altitude from A.

9.4 Construction of cyclic quadrilateral



If the vertices of a quadrilateral lie on a circle, then the quadrilateral is known as a cyclic quadrilateral. In a cyclic quadrilateral, the opposite angles are supplementary. That is, the sum of opposite angles is 180°. Thus, four suitable measurements (instead of five measurements) are sufficient for the construction of a cyclic quadrilateral.

Let us describe the various steps involved in the construction of a cyclic quadrilateral when the required measurements are given.



(i) Draw a rough figure and draw a $\triangle ABC$ or $\triangle ABD$ using the given measurements.

(ii) Draw the perpendicular bisectors of AB and BC intersecting each other at O. (one can take any two sides of $\triangle ABC$)

(iii) With O as the centre, and OA as radius, draw a circumcircle of $\triangle ABC$.

(iv) Using the given measurement, find the fourth vertex D and join AD and CD.

(v) Now, $ABCD$ is the required cyclic quadrilateral.

In this section, we shall construct a cyclic quadrilateral based on the different set of measurements of the cyclic quadrilateral as listed below.

(i) Three sides and one diagonal. (ii) Two sides and two diagonals. (iii) Three sides and one angle. (iv) Two sides and two angles. (v) One side and three angles. (vi) Two sides, one angle and one parallel line.

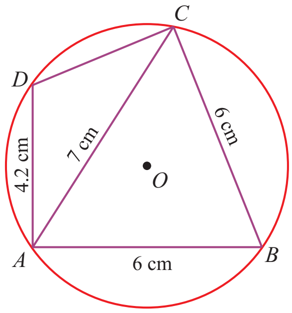
Type I (Three sides and one diagonal of a cyclic quadrilateral are given)

Example 9.7

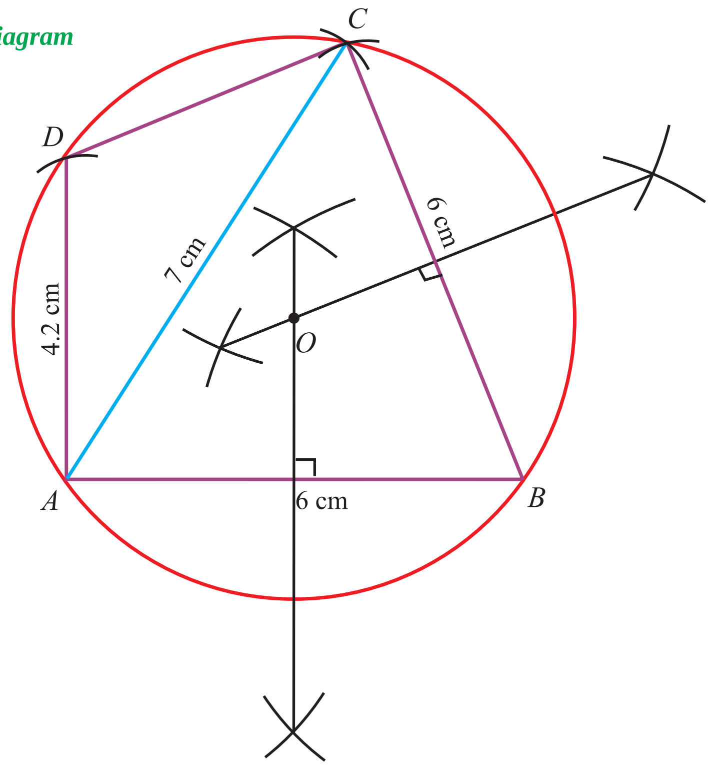
Construct a cyclic quadrilateral $ABCD$ in which $AB = 6$ cm, $AC = 7$ cm, $BC = 6$ cm, and $AD = 4.2$ cm.

Given : In the cyclic quadrilateral $ABCD$, $AB = 6$ cm, $AC = 7$ cm. $BC = 6$ cm, and $AD = 4.2$ cm.

Rough Diagram



Fair Diagram



Construction

(i) Draw a rough diagram and mark the measurements.
Draw a line segment $AB = 6$ cm.

(ii) With A and B as centres, draw arcs with radii 7 cm and 6 cm respectively, to intersect at C. Join AC and BC.

(iii) Draw the perpendicular bisectors of AB and BC to intersect at O.

(iv) With O as the centre and OA $(= OB = OC)$ as radius draw the circumcircle of ΔABC

(v) With A as the centre and radius 4.2 cm. draw an arc intersecting the circumcircle at D.

(vi) Join AD and CD.
Now, $ABCD$ is the required cyclic quadrilateral.

Type II (Two sides and two diagonals of a cyclic quadrilateral are given)

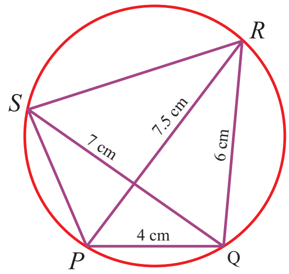
Example 9.8

Construct a cyclic quadrilateral $PQRS$ with $PQ = 4\text{cm}$, $QR = 6\text{cm}$, $PR = 7.5\text{ cm}$, $QS = 7\text{ cm}$

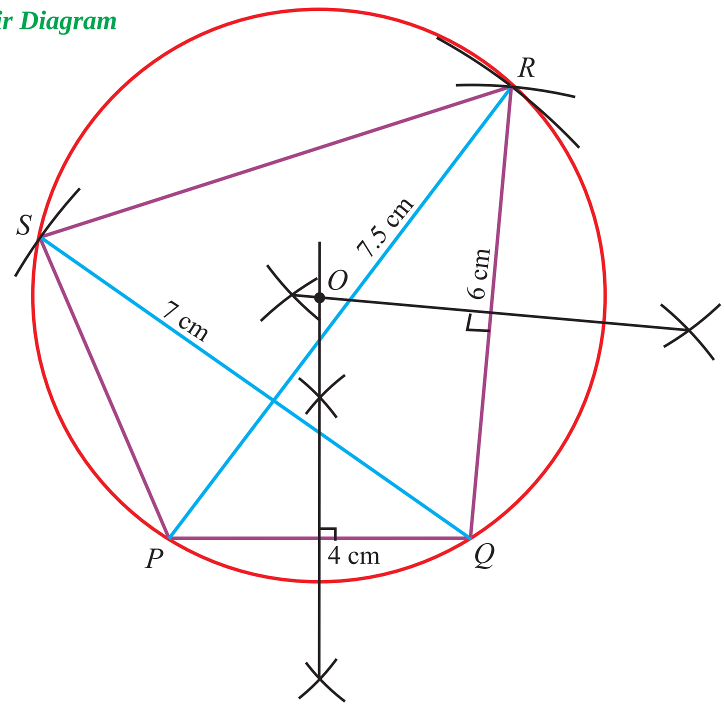
Given : In the cyclic quadrilateral $PQRS$, $PQ = 4\text{cm}$, $QR = 6\text{ cm}$,

$PR = 7.5\text{ cm}$ and $QS = 7\text{cm}$

Rough diagram



Fair Diagram



Construction

(i) Draw a rough diagram and mark the measurements.
Draw a line segment $PQ = 4\text{ cm}$

(ii) With P as centre and radius 7.5 cm, draw an arc.

(iii) With Q as centre and radius 6 cm, draw another arc meeting the previous arc as in the figure at R.

(iv) Join PR and QR.

(v) Draw the perpendicular bisectors of PQ and QR intersecting each other at O.

(vi) With O as the centre $OP(=OQ=OR)$ as radius, draw the circumcircle of ΔPQR.

(vii) With Q as centre and 7cm radius , draw an arc intersecting the circle at S.

(viii) Join PS and RS.

(ix) Now, $PQRS$ is the required cyclic quadrilateral.

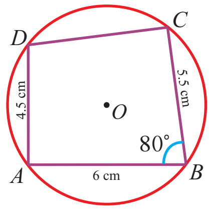
Type III (Three sides and one angle of a cyclic quadrilateral are given)

Example 9.9

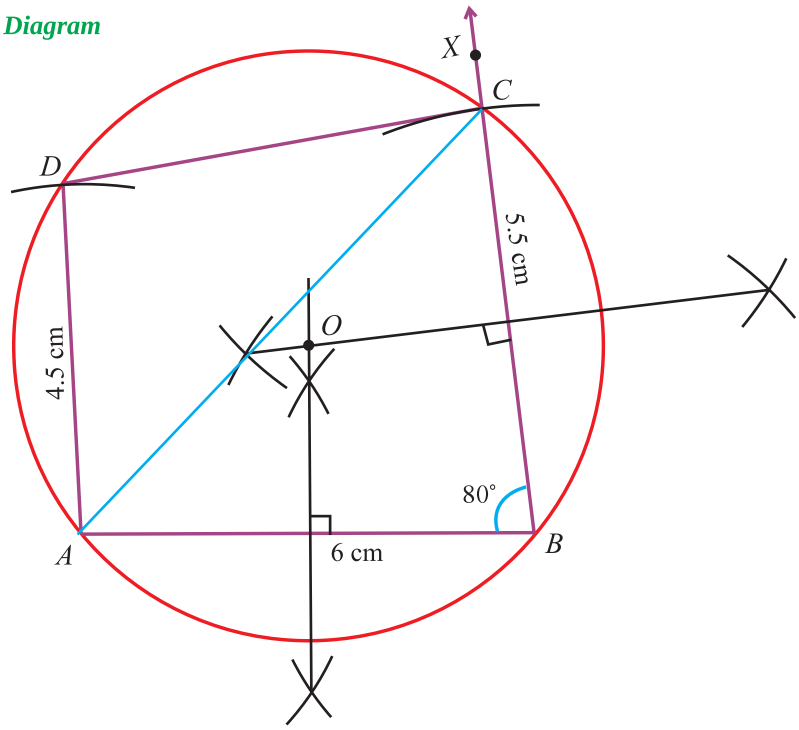
Construct a cyclic quadrilateral *ABCD* when $AB = 6$ cm, $BC = 5.5$ cm, $\angle ABC = 80^\circ$ and $AD = 4.5$ cm.

Given: In the Cyclic Quadrilateral *ABCD*, $AB = 6$ cm, $BC = 5.5$ cm, $\angle ABC = 80^\circ$ and $AD = 4.5$ cm.

Rough Diagram



Fair Diagram



Construction

(i) Draw a rough diagram and mark the measurements.

Draw a line segment $AB = 6$ cm.

(ii) Through *B* draw *BX* such that $\angle ABX = 80^\circ$.

(iii) With *B* as centre and radius 5.5 cm, draw an arc intersecting *BX* at *C* and join *AC*.

(iv) Draw the perpendicular bisectors of *AB* and *BC* intersecting each other at *O*.

(v) With *O* as centre and $OA\ (= OB = OC)$ as radius, draw the circumcircle of ΔABC.

(vi) With *A* as centre and radius 4.5 cm, draw an arc intersecting the circle at *D*.

(vii) Join *AD* and *CD*.

(viii) Now, *ABCD* is the required cyclic quadrilateral.

Type IV (Two sides and two angles of a cyclic quadrilateral are given)

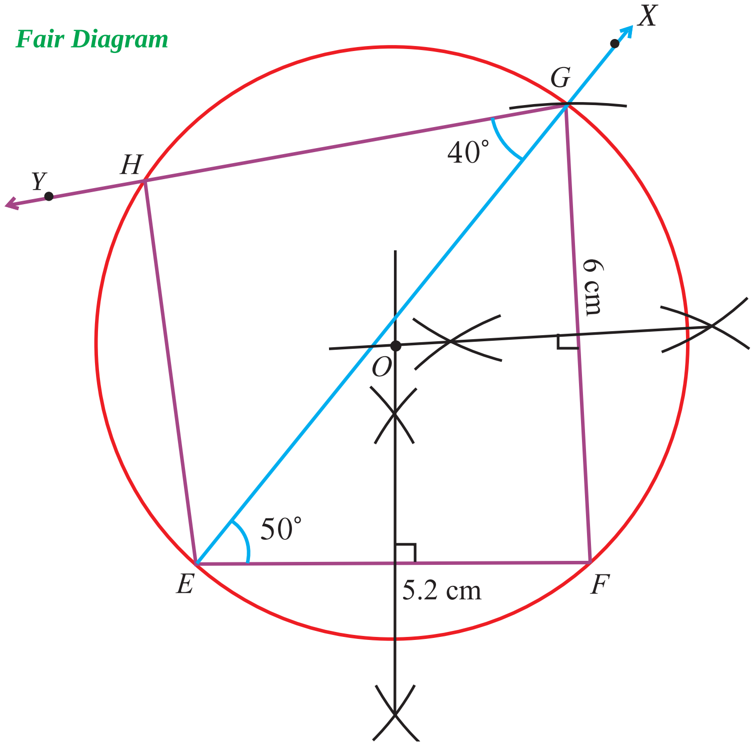
Example 9.10

Construct a cyclic quadrilateral $EFGH$ with $EF = 5.2$ cm, $\angle GEF = 50^\circ$, $FG = 6$ cm and $\angle EGH = 40^\circ$.

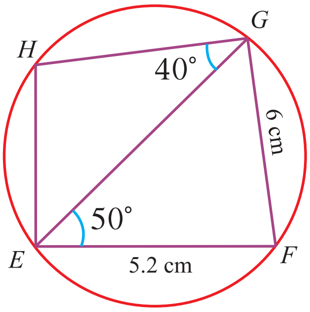
Given: In the Cyclic Quadrilateral $EFGH$

$EF = 5.2$ cm, $\angle GEF = 50^\circ$, $FG = 6$ cm and $\angle EGH = 40^o$.

Fair Diagram



Rough diagram



Construction

(i) Draw a rough diagram and mark the measurements.
Draw a line segment $EF = 5.2$ cm.

(ii) From E, draw EX such that $\angle FEX = 50^\circ$.

(iii) With F as centre and radius 6 cm, draw an arc intersecting EX at G.

(iv) Join FG.

(v) Draw the perpendicular bisectors of EF and FG intersecting each other at O.

(vi) With O as centre and OE $(= OF = OG)$ as radius, draw a circumcircle.

(vii) From G, draw GY such that $\angle EGY = 40^\circ$ which intersects the circle at H.

(viii) Join EH.

Now, $EFGH$ is the required cyclic quadrilateral.

Type V (One side and three angles of a cyclic quadrilateral are given)

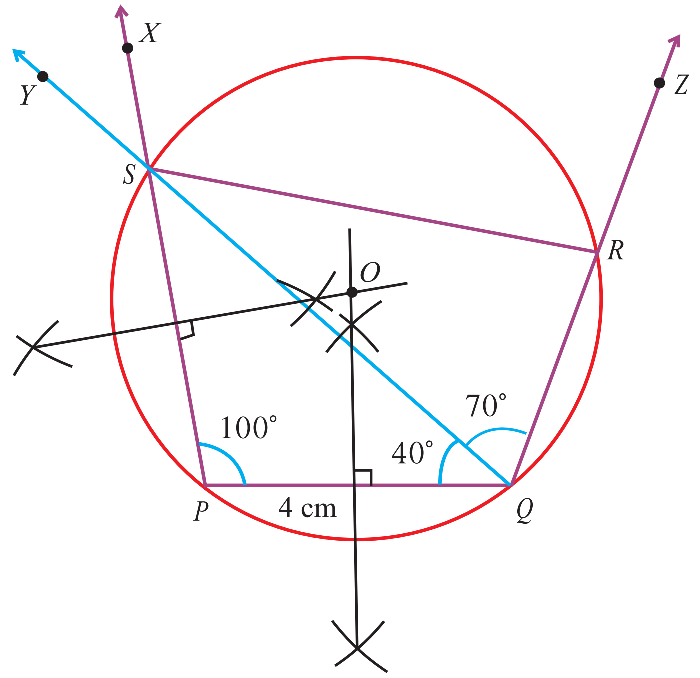
Example 9.11

Construct a cyclic quadrilateral $PQRS$ with $PQ = 4$ cm, $\angle P = 100^\circ$, $\angle PQS = 40^\circ$ and $\angle SQR = 70^\circ$.

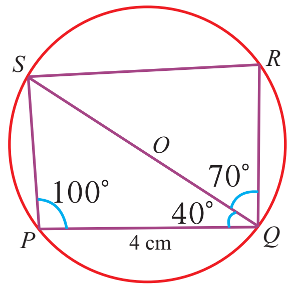
Given: In the cyclic quadrilateral $PQRS$,

$PQ = 4$ cm, $\angle P = 100^\circ$, $\angle PQS = 40^\circ$ and $\angle SQR = 70^\circ$.

Fair Diagram



Rough diagram



Construction

(i) Draw a rough diagram and mark the measurements.

Draw a line segment $PQ = 4$ cm.

(ii) From P draw PX such that $\angle QPX = 100^\circ$.

(iii) From Q draw QY such that $\angle PQY = 40^\circ$. Let QY meet PX at S.

(iv) Draw perpendicular bisectors of PQ and PS intersecting each other at O.

(v) With O as centre and $OP(= OQ = OS)$ as radius, draw a cicumcircle of $\triangle PQS$

(vi) From Q, draw QZ such that $\angle SQZ = 70^\circ$ which intersects the circle at R.

(vii) Join RS.

Now, $PQRS$ is the required cyclic quadrilateral.

Type VI

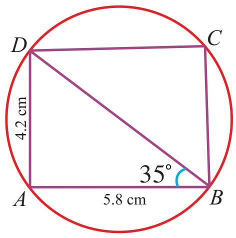
(Two sides , one angle and one parallel line are given)

Example 9.12

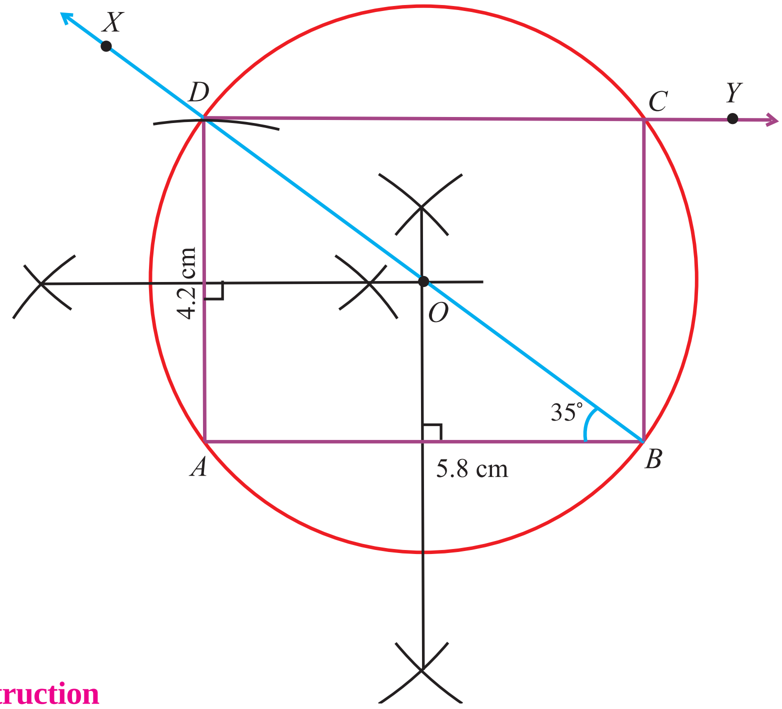
Construct a cyclic quadrilateral $ABCD$ when $AB = 5.8\,\text{cm}$, $\angle ABD = 35^\circ$, $AD = 4.2\,\text{cm}$ and $AB \parallel CD$.

Given: In the cyclic quadrilateral $ABCD$, $AB = 5.8\,\text{cm}$, $\angle ABD = 35^\circ$, $AD = 4.2\,\text{cm}$ and $AB \parallel CD$

Rough Diagram



Fair Diagram



Construction

(i) Draw a rough diagram and mark the measurements.

Draw a line segment $AB = 5.8\,\text{cm}$.

(ii) From B, draw BX such that $\angle ABX = 35^\circ$.

(iii) With A as centre and radius 4.2 cm, draw an arc intersecting BX at D.

(iv) Draw perpendicular bisectors of AB and AD intersecting each other at O.

(v) With O as centre, and $OA\ (= OB = OD)$ as radius, draw a circumcircle of ΔABD.

(vi) Draw DY such that $DY \parallel AB$ intersecting the circle at C.

Join BC.

(vii) Now, $ABCD$ is the required cyclic quadrilateral.

Exercise 9.3

1. Construct a cyclic quadrilateral $PQRS$, with $PQ = 6.5$cm, $QR = 5.5$ cm, $PR = 7$cm and $PS = 4.5$cm.

2. Construct a cyclic quadrilateral $ABCD$ where $AB = 6$cm, AD = 4.8cm, $BD = 8$cm and $CD = 5.5$ cm.

3. Construct a cyclic quadrilateral $PQRS$ such that $PQ = 5.5$cm, $QR = 4.5$cm, $\angle QPR = 45^\circ$ and $PS = 3$cm.

4. Construct a cyclic quadrilateral $ABCD$ with $AB = 7$cm, $\angle A = 80^\circ$, $AD = 4.5$ cm and $BC = 5$ cm.

5. Construct a cyclic quadrilateral $KLMN$ such that $KL = 5.5$ cm, $KM = 5$ cm, $LM = 4.2$ cm and $LN = 5.3$ cm.

6. Construct a cyclic quadrilateral $EFGH$ where $EF = 7$ cm, $EH = 4.8$ cm, $FH = 6.5$ cm and $EG = 6.6$ cm.

7. Construct a cyclic quadrilateral $ABCD$, given $AB = 6$ cm, $\angle ABC = 70^\circ$, $BC = 5$cm and $\angle ACD = 30^\circ$

8. Construct a cyclic quadrilateral $PQRS$ given $PQ = 5$ cm, $QR = 4$ cm, $\angle QPR = 35^\circ$ and $\angle PRS = 70^\circ$

9. Construct a cyclic quadrilateral $ABCD$ such that $AB = 5.5$ cm $\angle ABC = 50^\circ$, $\angle BAC = 60^\circ$ and $\angle ACD = 30^\circ$

10. Construct a cyclic quadrilateral $ABCD$, where $AB = 6.5$ cm, $\angle ABC = 110^\circ$, $BC = 5.5$ cm and $AB \parallel CD$.

Do you know?

Every year since 1901, the prestigious Nobel Prize has been awarded to individuals for achievements in Physics, Chemistry, Physiology or medicine, Literature and for Peace. The Nobel Prize is an international award administered by the Nobel Foundation in Stockholm, Sweden. There is no Nobel Prize for Mathematics.

The Fields medal is a prize awarded to two , three or four Mathematicians not over 40 years of age at each International congress of the International Mathematical Union (IMU), a meeting that takes place every four years.

The Fields medal is often described as the Nobel Prize for Mathematics.

10 GRAPHS

I think, therefore I am

- Rene Descartes

- Introduction
- Quadratic Graphs
- Special Graphs

Rene Descartes
(1596-1650)
France

Descartes devised the cartesian plane while he was in a hospital bed watching a fly buzzing around a corner of his room.

He created analytical geometry which paved the way of plotting graphs using coordinate axes.

10.1 Introduction

Graphs are diagrams that show information. They show how two different quantities are related to each other like weight is related to height. Sometimes algebra may be hard to visualize. Learning to show relationships between symbolic expressions and their graphs opens avenues to realize algebraic patterns.

Students should acquire the habit of drawing a reasonably accurate graph to illustrate a given problem under consideration. A carefully made graph not only serves to clarify the geometric interpretation of a problem but also may serve as a valuable check on the accuracy of the algebraic work. One should never forget that graphical results are at best only approximations, and of value only in proportion to the accuracy with which the graphs are drawn.

10.2 Quadratic Graphs

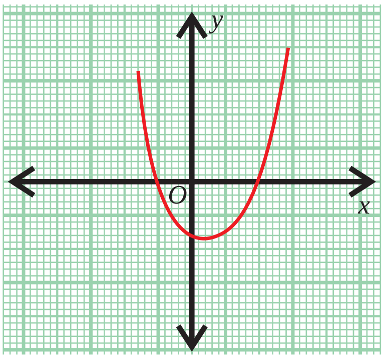
Definition

Let $f: A \to B$ be a function where A and B are subsets of $\mathbb{R}$. The set $\{(x,y) \mid x \in A, y = f(x)\}$ of all such ordered pairs (x, y) is called the graph of f.

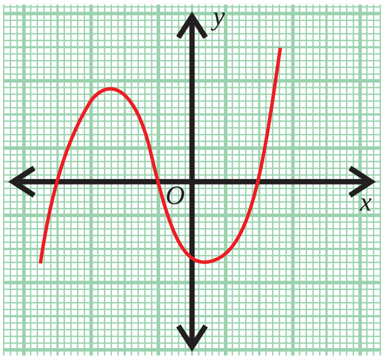
A polynomial function in x can be represented by a graph. The graph of a first degree polynomial $y = f(x) = ax + b, a \neq 0$ is an oblique line with slope a.

The graph of a second degree polynomial $y = f(x) = ax^2 + bx + c, \quad a \neq 0$ is a continuous non-linear curve, known as a parabola.

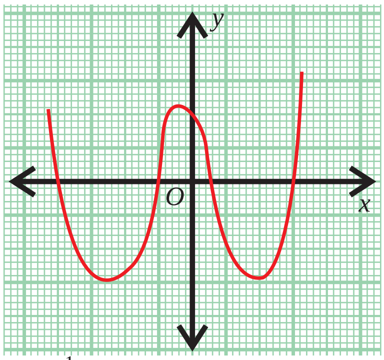
The following graphs represent different polynomials.



$y = (x+1)(x-2)$,
a polynomial of degree 2



$y = (x+4)(x+1)(x-2)$,
a polynomial of degree 3



$y = \frac{1}{14}(x+4)(x+1)(x-3)(x-0.5)$
a polynomial of degree 4

In class IX, we have learnt how to draw the graphs of linear polynomials of the form $y = ax + b, a \neq 0$. Now we shall focus on graphing a quadratic function $y = f(x) = ax^2 + bx + c$, where a, b and c are real constants, $a \neq 0$ and describe the nature of a quadratic graph.

Consider $y = ax^2 + bx + c$

By completing squares, the above polynomial can be rewritten as

$$\left(x + \frac{b}{2a}\right)^2 = \frac{1}{a}\left(y + \frac{b^2 - 4ac}{4a}\right).$$

Hence $\frac{1}{a}\left(y + \frac{b^2 - 4ac}{4a}\right) \geq 0$. (square of an expression is always positive)

The vertex of the curve (parabola) is $V\left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right)$

If $a > 0$, then the curve is open upward; it lies above or on the line $y = \frac{4ac - b^2}{4a}$ and it is symmetric about $x = -\frac{b}{2a}$.

If $a < 0$, then the curve is open downward; it lies below or on the line $y = \frac{4ac - b^2}{4a}$ and it is symmetric about $x = -\frac{b}{2a}$.

Let us give some examples of quadratic polynomials and the nature of their graphs in the following table.

S.No.	Polynomial ($y = ax^2 + bx + c$)	Vertex	Sign of a	Nature of curve
1	$y = 2x^2$ $a = 2, \ b = 0, \ c = 0$	(0, 0)	positive	(i) open upward (ii) lies above and on the line $y = 0$ (iii) symmetric about $x = 0$, i.e., y-axis
2	$y = -3x^2$ $a = -3, \ b = 0, \ c = 0$	(0, 0)	negative	(i) open downward (ii) lies below and on the line $y = 0$ (iii) symmetric about $x = 0$ i.e., y-axis
3	$y = x^2 - 2x - 3$ $a = 1, b = -2, c = -3$	(1, –4)	positive	(i) open upward (ii) lies above and on the line $y = -4$ (iii) symmetric about $x = 1$

Procedures to draw the quadratic graph $y = ax^2 + bx + c$

(i) Construct a table with the values of x and y using $y = ax^2 + bx + c$.

(ii) Choose a suitable scale.

The scale used on the x-axis does not have to be the same as the scale on the y-axis. The scale chosen should allow for the largest possible graph to be drawn. The bigger the graph, the more accurate will be the results obtained from it.

(iii) Plot the points on the graph paper and join these points by a smooth curve, as the graph of $y = ax^2 + bx + c$ does not contain line segments.

Example 10.1

Draw the graph of $y = 2x^2$.

Solution

First let us assign the integer values from -3 to 3 for x and form the following table.

x	-3	-2	-1	0	1	2	3
x^2	9	4	1	0	1	4	9
$y = 2x^2$	18	8	2	0	2	8	18

Plot the points $(-3, 18)$, $(-2, 8)$, $(-1, 2)$, $(0, 0)$, $(1, 2)$, $(2, 8)$, $(3, 18)$.

Join the points by a smooth curve.

The curve, thus obtained is the graph of $y = 2x^2$.

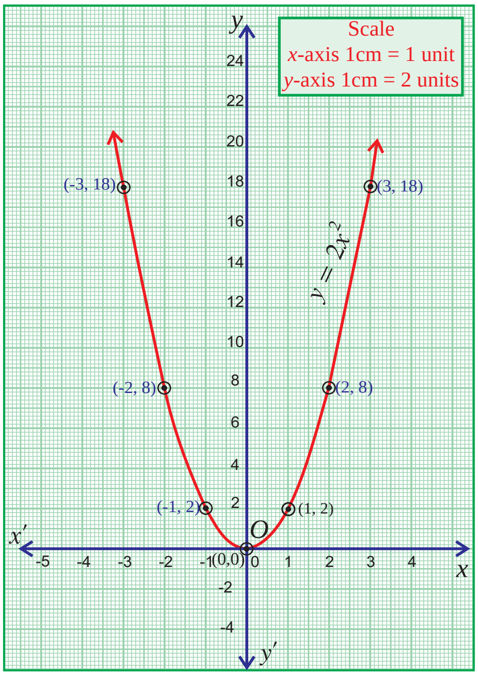


Fig. 10.1

Note

(i) It is symmetrical about y-axis. That is, the part of the graph to the left side of y-axis is the mirror image of the part to the right side of y-axis.

(ii) The graph does not lie below the x-axis as the values of y are non-negative.

Example 10.2

Draw the graph of $y = -3x^2$

Solution

Let us assign the integer values from -3 to 3 for x and form the following table.

x	-3	-2	-1	0	1	2	3
x^2	9	4	1	0	1	4	9
$y = -3x^2$	-27	-12	-3	0	-3	-12	-27

Plot the points $(-3, -27)$, $(-2, -12)$, $(-1, -3)$, $(0, 0)$, $(1, -3)$, $(2, -12)$ and $(3, -27)$.

Join the points by a smooth curve.

The curve thus obtained, is the graph of $y = -3x^2$

Note

(i) The graph of $y = -3x^2$ does not lie above the x-axis as y is always negative.

(ii) The graph is symmetrical about y-axis.

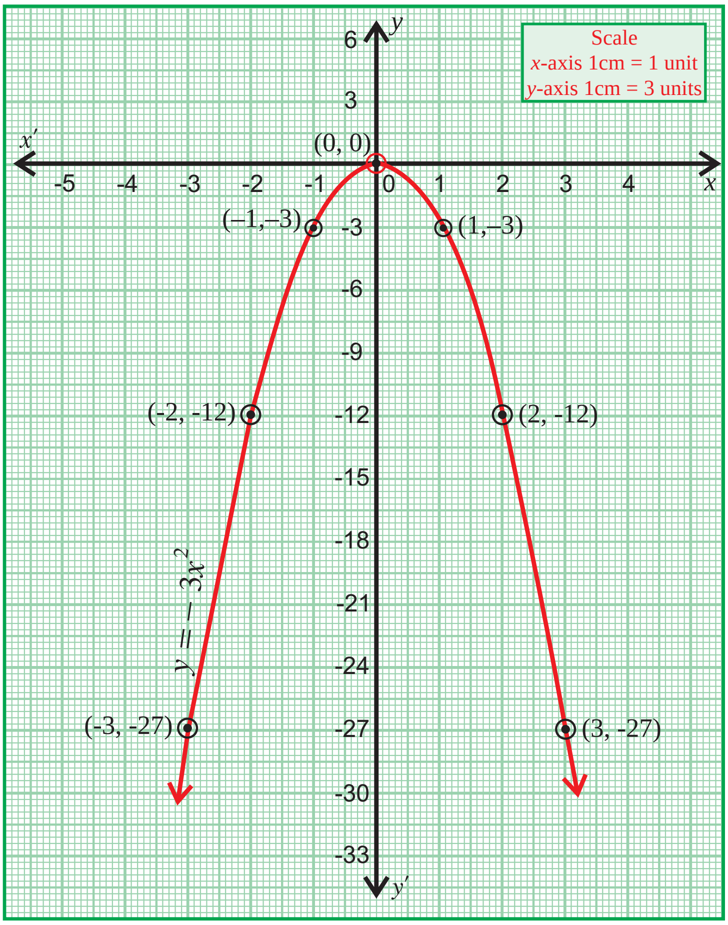


Fig. 10.2

10.2.1 To solve the quadratic equation $ax^2 + bx + c = 0$ graphically.

To find the roots of the quadratic equation $ax^2 + bx + c = 0$ graphically, let us draw the graph of $y = ax^2 + bx + c$. The x- coordinates of the points of intersection of the curve with the x-axis are the roots of the given equation, provided they intersect.

Example 10.3

Solve the equation $x^2 - 2x - 3 = 0$ graphically.

Solution

Let us draw the graph of $y = x^2 - 2x - 3$.

Now, form the following table by assigning integer values from –3 to 4 for x and finding the corresponding values of $y = x^2 - 2x - 3$.

x	–3	–2	–1	0	1	2	3	4
x^2	9	4	1	0	1	4	9	16
$-2x$	6	4	2	0	–2	–4	–6	–8
–3	–3	–3	–3	–3	–3	–3	–3	–3
y	12	5	0	–3	–4	–3	0	5

Plot the points (–3, 12), (–2, 5), (–1, 0), (0, –3), (1, –4), (2, –3), (3, 0), (4, 5) and join the points by a smooth curve.

The curve intersects the x-axis at the points (–1, 0) and (3, 0).

The x-coordinates of the above points are –1 and 3.

Hence, the solution set is {–1, 3}.

Note

(i) On the x-axis, $y = 0$ always.

(ii) The values of y are both positive and negative. Thus, the curve lies below and above the x-axis.

(iii) The curve is symmetric about the line $x = 1$. (It is not symmetric about the y-axis.)

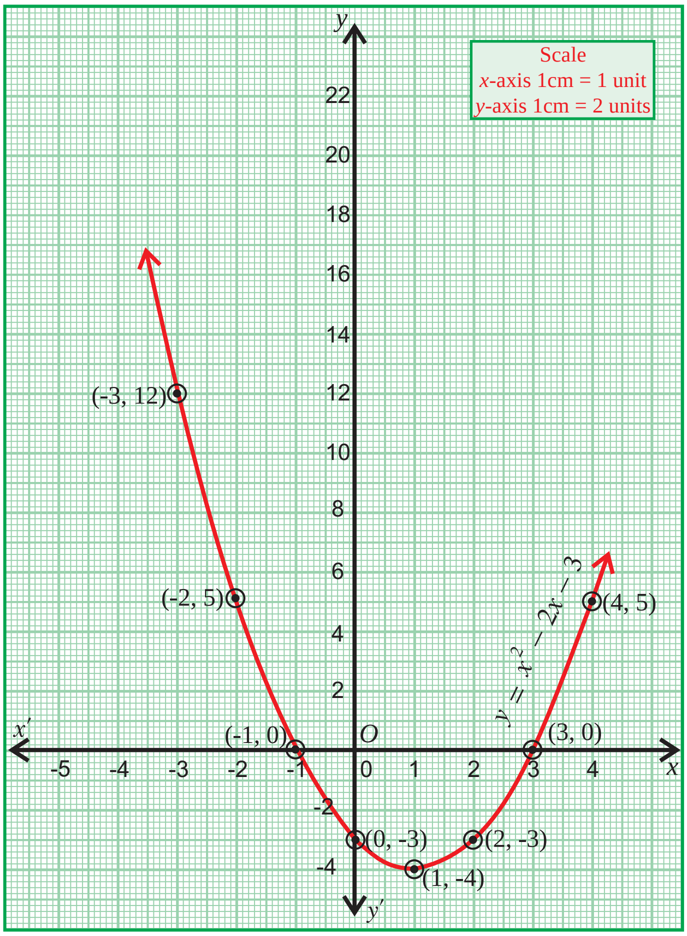


Fig. 10.3

Example 10.4

Solve graphically $2x^2 + x - 6 = 0$

Solution

First, let us form the following table by assigning integer values for x from –3 to 3 and finding the corresponding values of $y = 2x^2 + x - 6$.

x	–3	–2	–1	0	1	2	3
x^2	9	4	1	0	1	4	9
$2x^2$	18	8	2	0	2	8	18
x	–3	–2	–1	0	1	2	3
–6	–6	–6	–6	–6	–6	–6	–6
y	9	0	–5	–6	–3	4	15

Plot the points (–3, 9), (–2, 0), (–1, –5), (0, –6), (1, –3), (2, 4) and (3, 15) on the graph.

Join the points by a smooth curve. The curve, thus obtained, is the graph of $y = 2x^2 + x - 6$.

The curve cuts the x-axis at the points (–2 , 0) and (1.5 , 0).

The x-coordinates of the above points are –2 and 1.5.

Hence, the solution set is {–2 , 1.5 }.

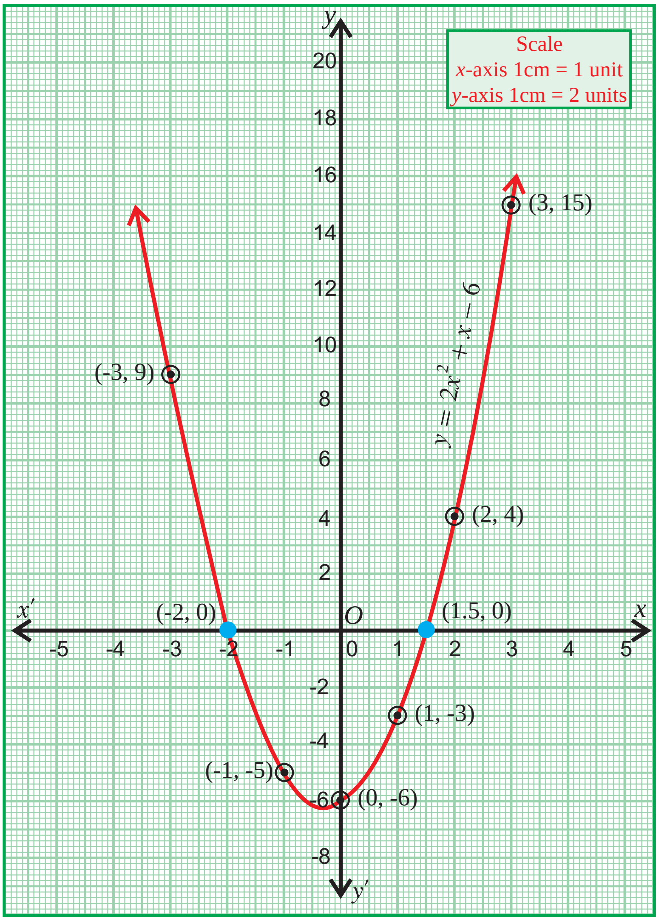


Fig. 10.4

Remarks

To solve $y = 2x^2 + x - 6$ graphically, one can proceed as follows.

(i) Draw the graph of $y = 2x^2$

(ii) Draw the graph of $y = 6 - x$

(iii) The x-coordinates of the points of intersection of the two graphs are the solutions of $2x^2 + x - 6 = 0$.

Example 10.5

Draw the graph of $y = 2x^2$ and hence solve $2x^2 + x - 6 = 0$.

Solution

First, let us draw the graph of $y = 2x^2$. Form the following table.

x	-3	-2	-1	0	1	2	3
x^2	9	4	1	0	1	4	9
$y = 2x^2$	18	8	2	0	2	8	18

Plot the points $(-3, 18)$, $(-2, 8)$, $(-1, 2)$, $(0, 0)$, $(1, 2)$, $(2, 8)$, $(3, 18)$.

Draw the graph by joining the points by a smooth curve.

To find the roots of $2x^2 + x - 6 = 0$, solve the two equations

$y = 2x^2$ and $2x^2 + x - 6 = 0$.

Now, $2x^2 + x - 6 = 0$.

$\Longrightarrow \quad y + x - 6 = 0$, since $y = 2x^2$

Thus, $y = -x + 6$

Hence, the roots of $2x^2 + x - 6 = 0$ are nothing but the x-coordinates of the points of intersection of

$y = 2x^2$ and $y = -x + 6$.

Now, for the straight line $y = -x + 6$, form the following table.

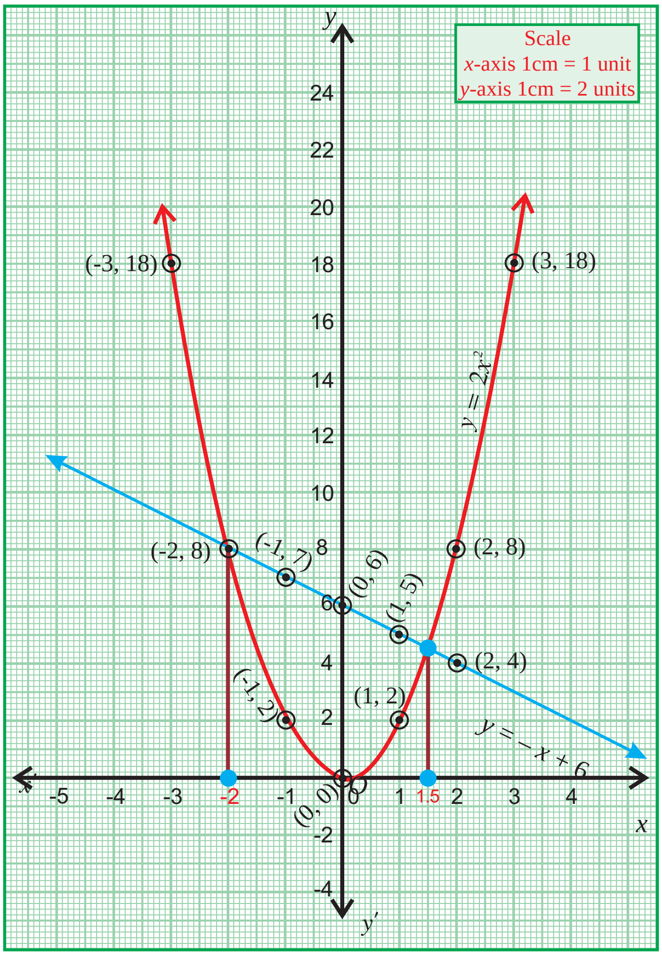


Fig. 10.5

x	-1	0	1	2
$y = -x + 6$	7	6	5	4

Draw the straight line by joining the above points.

The points of intersection of the line and the parabola are $(-2, 8)$ and $(1.5, 4.5)$. The x-coordinates of the points are -2 and 1.5.

Thus, the solution set for the equation $2x^2 + x - 6 = 0$ is $\{-2, 1.5\}$.

Example 10.6

Draw the graph of $y = x^2 + 3x + 2$ and use it to solve the equation $x^2 + 2x + 4 = 0$.

Solution

First, let us form a table for $y = x^2 + 3x + 2$.

x	–4	–3	–2	–1	0	1	2	3
x^2	16	9	4	1	0	1	4	9
$3x$	–12	–9	–6	–3	0	3	6	9
2	2	2	2	2	2	2	2	2
y	6	2	0	0	2	6	12	20

Plot the points (–4, 6), (–3, 2), (–2, 0), (–1, 0), (0, 2), (1, 6), (2, 12) and (3, 20).

Now, join the points by a smooth curve. The curve so obtained, is the graph of $y = x^2 + 3x + 2$.

Now, $x^2 + 2x + 4 = 0$

$\Longrightarrow$ $x^2 + 3x + 2 - x + 2 = 0$

$\Longrightarrow$ $y = x - 2$ $\quad \because y = x^2 + 3x + 2$

Thus, the roots of $x^2 + 2x + 4 = 0$ are obtained form the points of intersection of

$y = x - 2$ and $y = x^2 + 3x + 2$.

Let us draw the graph of the straight line $y = x - 2$.

Now, form the table for the line $y = x - 2$

x	–2	0	1	2
$y = x - 2$	–4	–2	–1	0

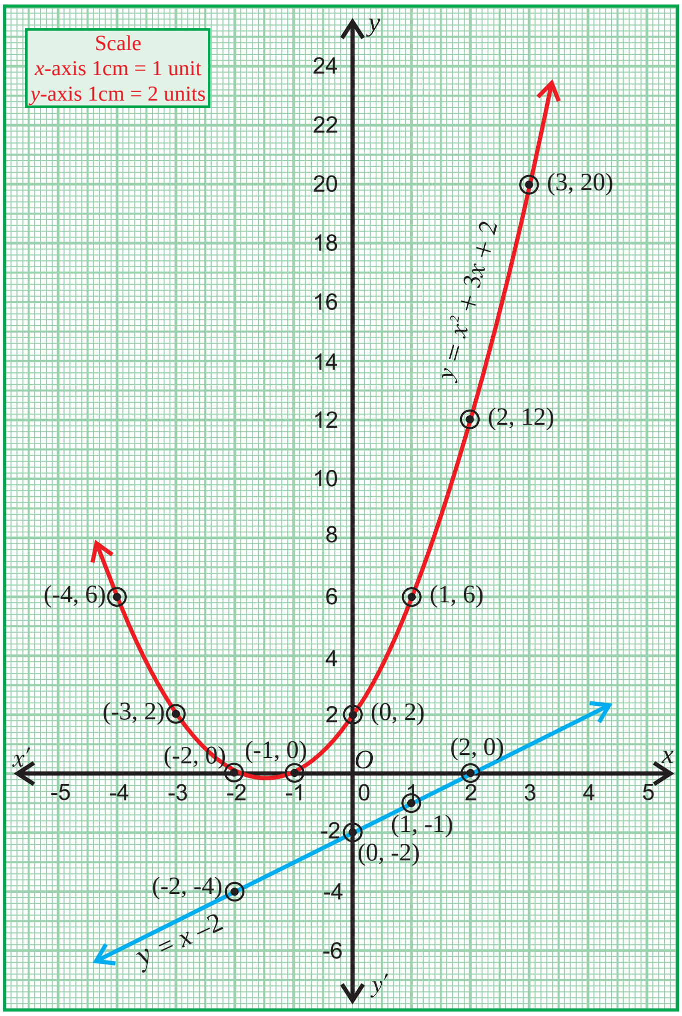


Fig. 10.6

The straight line $y = x - 2$ does not intersect the curve $y = x^2 + 3x + 2$.

Thus, $x^2 + 2x + 4 = 0$ has no real roots.

Exercise 10.1

1. Draw the graph of the following functions.

   (i) $y = 3x^2$

   (ii) $y = -4x^2$

   (iii) $y = (x+2)(x+4)$

   (iv) $y = 2x^2 - x + 3$

2. Solve the following equations graphically

   (i) $x^2 - 4 = 0$

   (ii) $x^2 - 3x - 10 = 0$

   (iii) $(x-5)(x-1) = 0$

   (iv) $(2x+1)(x-3) = 0$

3. Draw the graph of $y = x^2$ and hence solve $x^2 - 4x - 5 = 0$.
4. Draw the graph of $y = x^2 + 2x - 3$ and hence find the roots of $x^2 - x - 6 = 0$.
5. Draw the graph of $y = 2x^2 + x - 6$ and hence solve $2x^2 + x - 10 = 0$.
6. Draw the graph of $y = x^2 - x - 8$ and hence find the roots of $x^2 - 2x - 15 = 0$.
7. Draw the graph of $y = x^2 + x - 12$ and hence solve $x^2 + 2x + 2 = 0$.

10.3 Some Special Graphs

In this section, we will know how to draw graphs when the variables are in

(i) Direct variation (ii) Indirect variation.

If y is directly proportional to x, then we have $y = kx$, for some positive k. In this case the variables are said to be in direct variation and the graph is a straight line.

If y is inversely proportional to x, then we have $y = \frac{k}{x}$, for some positive k. In this case, the variables are said to be in indirect variation and the graph is a smooth curve , known as a Rectangular Hyperbola. (The equation of a rectangular hyperbola is of the form $xy = k,\ k > 0$.)

Example 10.7

Draw a graph for the following table and identify the variation.

x	2	3	5	8	10
y	8	12	20	32	40

Hence, find the value of y when $x = 4$.

Solution

From the table, we found that as x increases, y also increases. Thus, the variation is a direct variation.

Let $y = kx$.

$\Longrightarrow \quad \dfrac{y}{x} = k$

where k is the constant of proportionality.

From the given values, we have

$$k = \frac{8}{2} = \frac{12}{3} = \cdots = \frac{40}{10}. \quad \therefore \; k = 4$$

The relation $y = 4x$ forms a straight line graph.

Plot the points (2, 8), (3, 12), (5, 20), (8, 32) and (10, 40) and join these points to get the straight line.

Clearly, $y = 4x = 16$ when $x = 4$.

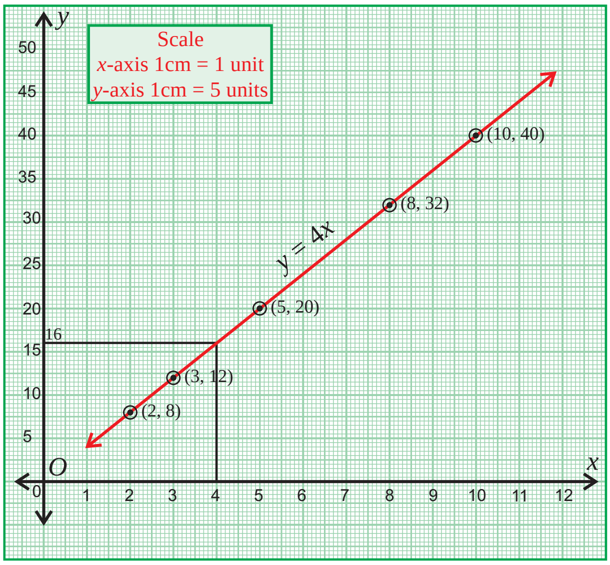


Fig. 10.7

Example 10.8

A cyclist travels from a place A to a place B along the same route at a uniform speed on different days. The following table gives the speed of his travel and the corresponding time he took to cover the distance.

Speed in km / hr x	2	4	6	10	12
Time in hrs y	60	30	20	12	10

Draw the speed-time graph and use it to find

(i) the number of hours he will take if he travels at a speed of 5 km / hr

(ii) the speed with which he should travel if he has to cover the distance in 40 hrs.

Solution

From the table, we observe that as x increases, y decreases.

This type of variation is called indirect variation.

Here, $xy = 120$.

Thus, $y = \frac{120}{x}$.

Plot the points (2 , 60), (4 , 30), (6 , 20), (10 , 12) and (12 , 10).

Join these points by a smooth curve.

From the graph, we have

(i) The number of hours he needed to travel at a speed of 5 km/hr is 24 hrs.

(ii) The required speed to cover the distance in 40 hrs, is 3 km / hr.

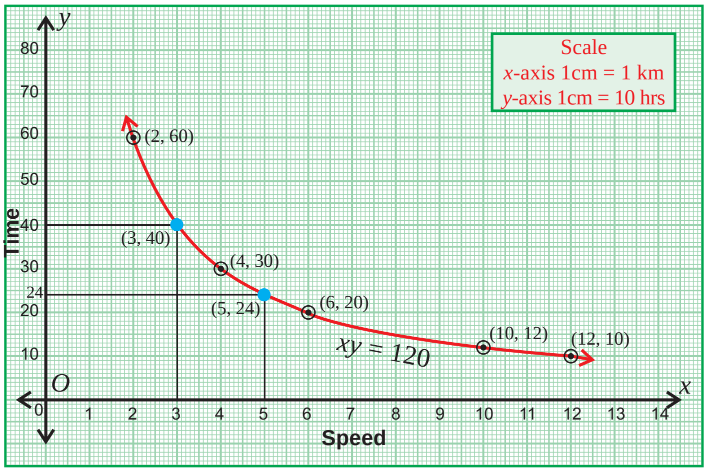


Fig. 10.8

Example 10.9

A bank gives 10% S.I on deposits for senior citizens. Draw the graph for the relation between the sum deposited and the interest earned for one year. Hence find

(i) the interest on the deposit of ₹650

(ii) the amount to be deposited to earn an interest of ₹45.

Solution

Let us form the following table.

Deposit ₹ x	100	200	300	400	500	600	700
S.I. earned ₹ y	10	20	30	40	50	60	70

Clearly $y = \frac{1}{10}x$ and the graph is a straight line.

Draw the graph using the points given in the table. From the graph, we see that

(i) The interest for the deposit of ₹650 is ₹65.

(ii) The amount to be deposited to earn an interest of ₹45 is ₹450.

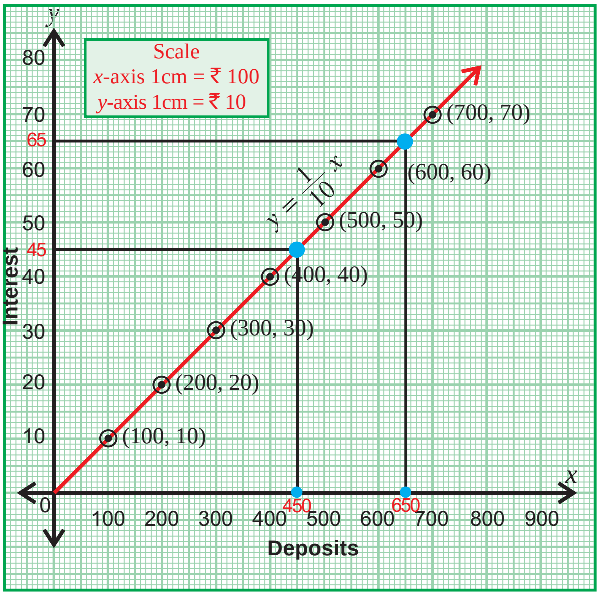


Fig. 10.9

Exercise 10.2

1. A bus travels at a speed of 40 km / hr. Write the distance-time formula and draw the graph of it. Hence, find the distance travelled in 3 hours.

2. The following table gives the cost and number of notebooks bought.

No. of note books x	2	4	6	8	10	12
Cost ₹ y	30	60	90	120	150	180

Draw the graph and hence (i) Find the cost of seven note books.

(ii) How many note books can be bought for ₹ 165.

3.

x	1	3	5	7	8
y	2	6	10	14	16

Draw the graph for the above table and hence find

(i) the value of y if $x = 4$

(ii) the value of x if $y = 12$

4. The cost of the milk per litre is ₹ 15. Draw the graph for the relation between the quantity and cost . Hence find

(i) the proportionality constant.

(ii) the cost of 3 litres of milk.

5. Draw the Graph of $xy = 20$, $x, y > 0$. Use the graph to find y when $x = 5$, and to find x when $y = 10$.

6.

No. of workers x	3	4	6	8	9	16
No of days y	96	72	48	36	32	18

Draw graph for the data given in the table. Hence find the number of days taken by 12 workers to complete the work.

Notable Quotes

1. In mathematics the art of proposing a question must be held of higher than solving it -Georg Cantor.
2. One reason why mathematics enjoys special esteem, above all other sciences, is that its laws are absolutely certain and indisputable, while those of other sciences are to some extent debatable and in constant danger of being overthrown by newly discovered facts - Albert Einstein

11

STATISTICS

It is easy to lie with statistics. It is hard to tell the truth without it
-Andrejs Dunkels

- Introduction
- Measures of Dispersion
  - Range
  - Variance
  - Standard Deviation
- Coefficient of Variation

Karl Pearson
(1857-1936)
England

Karl Pearson, British statistician, is a leading founder of modern field of statistics. He established the discipline of mathematical statistics. He introduced moments, a concept borrowed from physics.

His book, 'The Grammar of Science' covered several themes that were later to become part of the theories of Einstein and other scientists.

11.1 Introduction

According to Croxton and Cowden, Statistics is defined as the collection, presentation, analysis and interpretation of numerical data. R.A. Fisher said that the science of statistics is essentially a branch of Mathematics and may be regarded as mathematics applied to observational data. Horace Secrist defined statistics as follows:

"By statistics we mean aggregates of facts affected to a marked extent by multiplicity of causes, numerically expressed, enumerated or estimated according to reasonable standards of accuracy, collected in a systematic manner for a pre-determined purpose and placed in relation to each other".

The word 'Statistics' is known to have been used for the first time in " Elements of Universal Erudiation" by J.F. Baron. In modern times, statistics is no longer merely the collection of data and their presentation in charts and tables - it is now considered to encompass the science of basing inferences on observed data and the entire problem of making decisions in the face of uncertainity.

We have already learnt about the measures of central tendency namely, Mean, Median and Mode. They give us an idea of the concentration of the observation (data) about the central part of the distribution.

The knowledge of measures of central tendency cannot give a complete idea about the distribution. For example, consider the following two different series (i) 82, 74, 89, 95 and (ii) 120, 62, 28, 130. The two distributions have the same Mean 85. In the former, the numbers are closer to the mean 85 where as in the second series, the numbers are widely scattered about the Mean 85. Thus the measures of central tendency may mislead us. We need to have a measure which tells us how the items are dispersed around the Mean.

11.2 Measures of dispersion

Measures of dispersion give an idea about the scatteredness of the data of the distribution. Range (R), Quartile Deviation (Q.D), Mean Deviation (M.D) and Standard Deviation (S.D) are the measures of dispersion. Let us study about some of them in detail.

11.2.1 Range

Range is the simplest measure of dispersion. Range of a set of numbers is the difference between the largest and the smallest items of the set.

$\therefore$ Range = Largest value $-$ Smallest value

$= L - S.$

The coefficient of range is given by $\dfrac{L-S}{L+S}$

Example 11.1

Find the range and the coefficient of range of 43, 24, 38, 56, 22, 39, 45.

Solution Let us arrange the given data in the ascending order.

22, 24, 38, 39, 43, 45, 56.

From the given data the largest value, $L = 56$ and the smallest value, $S = 22$.

$$\therefore \quad \text{Range} = L - S = 56 - 22 = 34$$

$$\text{Now the coefficient of range} = \frac{L-S}{L+S} = \frac{56-22}{56+22} = \frac{34}{78} = 0.436$$

Example 11.2

The weight (in kg) of 13 students in a class are 42.5, 47.5, 48.6, 50.5, 49, 46.2, 49.8, 45.8, 43.2, 48, 44.7, 46.9, 42.4. Find the range and coefficient of range.

Solution Let us arrange the given data in the ascending order.

42.4, 42.5, 43.2, 44.7, 45.8, 46.2, 46.9, 47.5, 48, 48.6, 49, 49.8, 50.5

From the given data, the largest value $L = 50.5$ and the smallest value $S = 42.4$

$$\text{Range} = L - S = 50.5 - 42.4 = 8.1$$

$$\text{The coefficient of range} = \frac{L-S}{L+S} = \frac{50.5-42.4}{50.5+42.4} = \frac{8.1}{92.9} = 0.087$$

Example 11.3

The largest value in a collection of data is 7.44. If the range is 2.26, then find the smallest value in the collection.

Solution Range = largest value – smallest value

$\Longrightarrow$ 7.44 – smallest value = 2.26

$\therefore$ The smallest value = 7.44–2.26 = 5.18

11.2.2 Standard deviation

A better way to measure dispersion is to square the differences between each data and the mean before averaging them. This measure of dispersion is known as the Variance and the positive square root of the Variance is known as the Standard Deviation. The variance is always positive.

The term 'standard deviation' was first used by Karl Pearson in 1894 as a replacement of the term 'mean error' used by Gauss.

Standard deviation is expressed in the same units as the data. It shows how much variation is there from the mean. A low standard deviation indicates that the data points tend to be very close to the mean, where as a high standard deviation indicates that the data is spread out over a large range of values.

We use $\overline{x}$ and σ to denote the mean and the standard deviation of a distribution respectively. Depending on the nature of data, we shall calculate the standard deviation σ (after arranging the given data either in ascending or descending order) by different methods using the following formulae (proofs are not given).

Data	Direct method	Actual mean method	Assumed mean method	Step deviation method
Ungrouped	$\sqrt{\frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2}$	$\sqrt{\frac{\sum d^2}{n}}$ $d = x - \overline{x}$	$\sqrt{\frac{\sum d^2}{n} - \left(\frac{\sum d}{n}\right)^2}$ $d = x - A$	$\sqrt{\frac{\sum d^2}{n} - \left(\frac{\sum d}{n}\right)^2} \times c$ $d = \frac{x - A}{c}$
Grouped		$\sqrt{\frac{\sum fd^2}{\sum f}}$	$\sqrt{\frac{\sum fd^2}{\sum f} - \left(\frac{\sum fd}{\sum f}\right)^2}$	$\sqrt{\frac{\sum fd^2}{\sum f} - \left(\frac{\sum fd}{\sum f}\right)^2} \times c$

Note

For a collection of n items (numbers), we always have

$\sum(x - \overline{x}) = 0$, $\sum x = n\overline{x}$ and $\sum \overline{x} = n\overline{x}$.

(i) Direct method

This method can be used, when the squares of the items are easily obtained.

Example 11.4

The number of books read by 8 students during a month are

2, 5, 8, 11, 14, 6, 12, 10. Calculate the standard deviation of the data.

Solution

x	x^2
2	4
5	25
6	36
8	64
10	100
11	121
12	144
14	196
$\sum x=68$	$\sum x^2=690$

Here, the number of items, $n = 8$

Standard deviation, $\sigma = \sqrt{\frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2}$

$= \sqrt{\frac{690}{8} - \left(\frac{68}{8}\right)^2}$

$= \sqrt{86.25 - (8.5)^2}$

$= \sqrt{86.25 - 72.25}$

$= \sqrt{14} \simeq 3.74$

(ii) Actual mean method

This method can be used when the mean is not a fraction.

The standard deviation, $\sigma = \sqrt{\frac{\sum (x-\overline{x})^2}{n}}$ or $\sigma = \sqrt{\frac{\sum d^2}{n}}$, where $d = x - \overline{x}$.

Example 11.5

A test in General Knowledge was conducted for a class. The marks out of 40, obtained by 6 students were 20, 14, 16, 30, 21 and 25. Find the standard deviation of the data.

Solution Now, A. M. $= \frac{\sum x}{n} = \frac{20+14+16+30+21+25}{6}$

$\Longrightarrow \quad \overline{x} = \frac{126}{6} = 21.$

Let us form the following table.

x	$d = x - \overline{x}$	d^2
14	-7	49
16	-5	25
20	-1	1
$\boxed{21}$	0	0
25	4	16
30	9	81
$\sum x = 126$	$\sum d = 0$	$\sum d^2 = 172$

$\sigma = \sqrt{\frac{\sum d^2}{n}} = \sqrt{\frac{172}{6}}$

$= \sqrt{28.67}$

Thus, $\sigma \simeq 5.36$

(iii) Assumed mean method

When the mean of the given data is not an integer, we use assumed mean method to calculate the standard deviation. We choose a suitable item A such that the difference $x–A$ are all small numbers possibly, integers. Here A is an assumed mean which is supposed to be closer to the mean.

We calculate the deviations using $d = x - A$.

Now the standard deviation, $\sigma = \sqrt{\frac{\sum d^2}{n} - \left(\frac{\sum d}{n}\right)^2}$.

Note

Assumed mean method and step deviation method are just simplified forms of direct method.

Example: 11.6

Find the standard deviation of the numbers 62, 58, 53, 50, 63, 52, 55.

Solution Let us take $A = 55$ as the assumed mean and form the following table.

x	$d = x - A$ $= x - 55$	d^2
50	-5	25
52	-3	9
53	-2	4
55	0	0
58	3	9
62	7	49
63	8	64
	$\sum d = 8$	$\sum d^2 = 160$

$$\sigma = \sqrt{\frac{\sum d^2}{n} - \left(\frac{\sum d}{n}\right)^2}$$

$$= \sqrt{\frac{160}{7} - \left(\frac{8}{7}\right)^2}$$

$$= \sqrt{\frac{160}{7} - \frac{64}{49}}$$

$$= \sqrt{\frac{1056}{49}}$$

$$= \frac{32.49}{7}$$

$\therefore$ Standard deviation $\sigma \simeq 4.64$

(iv) Step deviation method

This method can be used to find the standard deviation when the items are larger in size and have a common factor. We choose an assumed mean A and calculate d *by* using $d = \frac{x - A}{c}$ where c is the common factor of the values of $x–A$.

We use the formula, $\sigma = \sqrt{\frac{\sum d^2}{n} - \left(\frac{\sum d}{n}\right)^2} \times c.$

Example 11.7

The marks obtained by 10 students in a test in Mathematics are :

80, 70, 40, 50, 90, 60, 100, 60, 30, 80. Find the standard deviation.

Solution We observe that all the data have 10 as common factor. Take A = 70 as assumed mean. Here the number of items, n = 10.

Take $c = 10$, $d = \frac{x - A}{10}$ and form the following table.

x	$d = \frac{x-70}{10}$	d^2
30	-4	16
40	-3	9
50	-2	4
60	-1	1
60	-1	1
$\boxed{70}$	0	0
80	1	1
80	1	1
90	2	4
100	3	9
	$\sum d = -4$	$\sum d^2 = 46$

Now $\quad \sigma = \sqrt{\frac{\sum d^2}{n} - \left(\frac{\sum d}{n}\right)^2} \times c$

$$= \sqrt{\frac{46}{10} - \left(\frac{-4}{10}\right)^2} \times 10$$

$$= \sqrt{\frac{46}{10} - \frac{16}{100}} \times 10 = \sqrt{\frac{460-16}{100}} \times 10$$

$\therefore$ Standard deviation, $\sigma \simeq 21.07$

The standard deviation for a collection of data can be obtained in any of the four methods, namely direct method, actual mean method, assumed mean method and step deviation method.

As expected, the different methods should not give different answers for σ for the same data. Students are advised to follow any one of the above methods.

Results

(i) The standard deviation of a distribution remains unchanged when each value is added or subtracted by the same quantity.

(ii) If each value of a collection of data is multiplied or divided by a non-zero constant k, then the standard deviation of the new data is obtained by multiplying or dividing the standard deviation by the same quantity k.

Example: 11.8

Find the standard deviation of the data 3, 5, 6, 7. Then add 4 to each item and find the standard deviation of the new data.

Solution Given data 3, 5, 6, 7

Take $A = 6$

x	$d = x - 6$	d^2
3	-3	9
5	-1	1
$\boxed{6}$	0	0
7	1	1
	$\sum d = -3$	$\sum d^2 = 11$

Standard deviation, $\sigma = \sqrt{\frac{\sum d^2}{n} - \left(\frac{\sum d}{n}\right)^2}$

$= \sqrt{\frac{11}{4} - \left(\frac{-3}{4}\right)^2}$

$\sigma = \sqrt{\frac{11}{4} - \frac{9}{16}} = \frac{\sqrt{35}}{4}$

Let us add 4 to each term of the given data to get the new data 7, 9, 10, 11

Take $A = 10$

x	$d = x - 10$	d^2
7	-3	9
9	-1	1
$\boxed{10}$	0	0
11	1	1
	$\sum d = -3$	$\sum d^2 = 11$

Standard deviation, $\sigma_1 = \sqrt{\frac{\sum d^2}{n} - \left(\frac{\sum d}{n}\right)^2}$

$= \sqrt{\frac{11}{4} - \left(\frac{-3}{4}\right)^2}$

$\sigma_1 = \sqrt{\frac{11}{4} - \frac{9}{16}} = \frac{\sqrt{35}}{4}$

In the above example, the standard deviation remains unchanged even when each item is added by the constant 4.

Example 11.9

Find the standard deviation of 40, 42 and 48. If each value is multiplied by 3, find the standard deviation of the new data.

Solution Let us consider the given data 40, 42, 48 and find σ.

Let the assumed mean A be 44

x	$d = x - 44$	d^2
40	-4	16
42	-2	4
48	4	16
	$\sum d = -2$	$\sum d^2 = 36$

Standard deviation, $\sigma = \sqrt{\frac{\sum d^2}{n} - \left(\frac{\sum d}{n}\right)^2}$

$= \sqrt{\frac{36}{3} - \left(\frac{-2}{3}\right)^2}$

$\sigma = \frac{\sqrt{104}}{3}$

When the values are multiplied by 3, we get 120,126,144. Let the assumed mean A be 132.

Let σ_1 be the S.D. of the new data.

x	$d = x - 132$	d^2
120	-12	144
126	-6	36
144	12	144
	$\sum d = -6$	$\sum d^2 = 324$

Standard deviation, $\sigma_1 = \sqrt{\frac{\sum d^2}{n} - \left(\frac{\sum d}{n}\right)^2}$

$= \sqrt{\frac{324}{3} - \left(\frac{-6}{3}\right)^2}$

$\sigma_1 = \sqrt{\frac{312}{3}} = \sqrt{104}$

In the above example, when each value is multiplied by 3, the standard deviation also gets multiplied by 3 .

Example 11.10

Prove that the standard deviation of the first n natural numbers is $\sigma = \sqrt{\dfrac{n^2-1}{12}}$.

Solution The first n natural numbers are 1,2,3, $\cdots$, n.

$$\text{Their mean, } \quad \overline{x} = \frac{\Sigma x}{n} = \frac{1+2+3+\cdots+n}{n}$$

$$= \frac{n(n+1)}{2n} = \frac{n+1}{2}.$$

Sum of the squares of the first n natural numbers is

$$\sum x^2 = \frac{n(n+1)(2n+1)}{6}.$$

Thus, the standard deviation $\sigma = \sqrt{\dfrac{\sum x^2}{n} - \left(\dfrac{\sum x}{n}\right)^2}$

$$= \sqrt{\frac{n(n+1)(2n+1)}{6n} - \left(\frac{n+1}{2}\right)^2}$$

$$= \sqrt{\frac{(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2}$$

$$= \sqrt{\left(\frac{n+1}{2}\right)\left[\frac{(2n+1)}{3} - \frac{(n+1)}{2}\right]}$$

$$= \sqrt{\left(\frac{n+1}{2}\right)\left[\frac{2(2n+1)-3(n+1)}{6}\right]}$$

$$= \sqrt{\left(\frac{n+1}{2}\right)\left(\frac{4n+2-3n-3}{6}\right)}$$

$$= \sqrt{\left(\frac{n+1}{2}\right)\left(\frac{n-1}{6}\right)}$$

$$= \sqrt{\frac{n^2-1}{12}}.$$

Hence, the S.D. of the first n natural numbers is $\sigma = \sqrt{\dfrac{n^2-1}{12}}$.

Remarks

It is quite interesting to note the following:

The S.D. of any n successive terms of an A.P. with common difference d is, $\sigma = d\sqrt{\dfrac{n^2-1}{12}}$.

Thus,

(i) S.D. of $i, i+1, i+2, \cdots, i+n$ is $\sigma = \sqrt{\dfrac{n^2-1}{12}}$, $i \in \mathbb{N}$.

(ii) S.D. of any n consecutive even integers, is given by $\sigma = 2\sqrt{\dfrac{n^2-1}{12}}$, $n \in \mathbb{N}$.

(iii) S.D. of any n consecutive odd integers, is given by $\sigma = 2\sqrt{\dfrac{n^2-1}{12}}$, $n \in \mathbb{N}$.

Example 11.11

Find the standard deviation of the first 10 natural numbers.

Solution Standard deviation of the first n natural numbers = $\sqrt{\frac{n^2-1}{12}}$

Standard deviation of the first 10 natural numbers

$$= \sqrt{\frac{10^2-1}{12}} = \sqrt{\frac{100-1}{12}} \simeq 2.87$$

Standard Deviation of grouped data

(i) Actual mean method

In a discrete data, when the deviations are taken from arithmetic mean, the standard deviation can be calculated using the formula $\sigma = \sqrt{\frac{\sum fd^2}{\sum f}}$, where $d = x - \overline{x}$.

Example 11.12

The following table shows the marks obtained by 48 students in a Quiz competition in Mathematics. Calculate the standard deviation.

Data x	6	7	8	9	10	11	12
Frequency f	3	6	9	13	8	5	4

Solution Let us form the following table using the given data.

x	f	fx	$d = x - \overline{x}$ $= x - 9$	fd	fd^2
6	3	18	-3	-9	27
7	6	42	-2	-12	24
8	9	72	-1	-9	9
$\boxed{9}$	13	117	0	0	0
10	8	80	1	8	8
11	5	55	2	10	20
12	4	48	3	12	36
	$\sum f = 48$	$\sum fx = 432$	$\sum d = 0$	$\sum fd = 0$	$\sum fd^2 = 124$

Arithmetic mean, $\bar{x} = \frac{\sum fx}{\sum f} = \frac{432}{48} = 9.$

Standard deviation, $\sigma = \sqrt{\frac{\sum fd^2}{\sum f}}$

$$= \sqrt{\frac{124}{48}}$$

$$\sigma = \sqrt{2.58} \simeq 1.61$$

(ii) Assumed mean method

When deviations are taken from the assumed mean, the formula for calculating standard deviation is

$$\sigma = \sqrt{\frac{\Sigma fd^2}{\Sigma f} - \left(\frac{\Sigma fd}{\Sigma f}\right)^2}, \text{ where } d = x - A.$$

Example 11.13

Find the standard deviation of the following distribution.

x	70	74	78	82	86	90
f	1	3	5	7	8	12

Solution Let us take the assumed mean $A = 82$.

x	f	$d = x - 82$	fd	fd^2
70	1	-12	-12	144
74	3	-8	-24	192
78	5	-4	-20	80
$\boxed{82}$	7	0	0	0
86	8	4	32	128
90	12	8	96	768
	$\sum f = 36$		$\sum fd = 72$	$\sum fd^2 = 1312$

$$\text{Standard deviation } \sigma = \sqrt{\frac{\sum fd^2}{\sum f} - \left(\frac{\sum fd}{\sum f}\right)^2}$$

$$= \sqrt{\frac{1312}{36} - \left(\frac{72}{36}\right)^2}$$

$$= \sqrt{\frac{328}{9} - 2^2}$$

$$= \sqrt{\frac{328 - 36}{9}}$$

$$= \sqrt{\frac{292}{9}} = \sqrt{32.44}$$

$$\therefore \quad \sigma \simeq 5.7$$

Example 11.14

Find the variance of the following distribution.

Class interval	3.5-4.5	4.5-5.5	5.5-6.5	6.5-7.5	7.5-8.5
Frequency	9	14	22	11	17

Solution Let the assumed mean A be 6.

class interval	x mid value	f	$d = x - 6$	fd	fd^2
3.5-4.5	4	9	-2	-18	36
4.5-5.5	5	14	-1	-14	14
5.5-6.5	$\boxed{6}$	22	0	0	0
6.5-7.5	7	11	1	11	11
7.5-8.5	8	17	2	34	68
		$\sum f = 73$		$\sum fd = 13$	$\sum fd^2 = 129$

Now variance, $\sigma^2 = \dfrac{\sum fd^2}{\sum f} - \left(\dfrac{\sum fd}{\sum f}\right)^2$

$$= \frac{129}{73} - \left(\frac{13}{73}\right)^2 = \frac{129}{73} - \frac{169}{5329}$$

$$= \frac{9417 - 169}{5329} = \frac{9248}{5329}$$

Thus, the variance is $\sigma^2 \simeq 1.74$

(iii) Step deviation method

Example 11.15

The following table gives the number of goals scored by 71 leading players in International Football matches. Find the standard deviation of the data.

Class Interval	0-10	10-20	20-30	30-40	40-50	50-60	60-70
Frequency	8	12	17	14	9	7	4

Solution Let $A = 35$. In the 4th column, the common factor of all items, $c = 10$.

class interval	x mid value	f	$x–A$	$d = \dfrac{x-A}{c}$	fd	fd^2
0 – 10	5	8	–30	-3	-24	72
10 – 20	15	12	–20	-2	-24	48
20 – 30	25	17	–10	-1	-17	17
30 – 40	$\boxed{35}$	14	0	0	0	0
40 – 50	45	9	10	1	9	9
50 – 60	55	7	20	2	14	28
60 – 70	65	4	30	3	12	36
		$\sum f = 71$			$\sum fd = -30$	$\sum fd^2 = 210$

Standard deviation, $\sigma = \sqrt{\frac{\sum fd^2}{\sum f} - \left(\frac{\sum fd}{\sum f}\right)^2} \times c$

$= \sqrt{\frac{210}{71} - \left(\frac{-30}{71}\right)^2} \times 10$

$= \sqrt{\frac{210}{71} - \frac{900}{5041}} \times 10$

$= \sqrt{\frac{14910 - 900}{5041}} \times 10$

$= \sqrt{\frac{14010}{5041}} \times 10 = \sqrt{2.7792} \times 10$

Standard deviation, $\sigma \simeq 16.67$

Example 11.16

Length of 40 bits of wire, correct to the nearest centimetre are given below. Calculate the variance.

Length cm	1-10	11-20	21-30	31-40	41-50	51-60	61-70
No. of bits	2	3	8	12	9	5	1

Solution Let the assumed mean A be 35.5

Length	mid value x	no. of bits (f)	$d = x - A$	fd	fd^2
1-10	5.5	2	-30	-60	1800
11-20	15.5	3	-20	-60	1200
21-30	25.5	8	-10	-80	800
31-40	$\boxed{35.5}$	12	0	0	0
41-50	45.5	9	10	90	900
51-60	55.5	5	20	100	2000
61-70	65.5	1	30	30	900
		$\sum f = 40$		$\sum fd = 20$	$\sum fd^2 = 7600$

Variance, $\sigma^2 = \frac{\sum fd^2}{\sum f} - \left(\frac{\sum fd}{\sum f}\right)^2 = \frac{7600}{40} - \left(\frac{20}{40}\right)^2$

$= 190 - \frac{1}{4} = \frac{760 - 1}{4} = \frac{759}{4}$

$\therefore \quad \sigma^2 = 189.75$

11.2.3 Coefficient of variation

Coefficient of variation is defined as

$$\text{C.V} = \frac{\sigma}{\overline{x}} \times 100$$

where σ is the standard deviation and $\overline{x}$ is the mean of the given data. It is also called as a relative standard deviation.

Remarks

(i) The coefficient of variation helps us to compare the consistency of two or more collections of data.

(ii) When the coefficient of variation is more, the given data is less consistent.

(iii) When the coefficient of variation is less, the given data is more consistent.

Example 11.17

Find the coefficient of variation of the following data. 18, 20, 15, 12, 25.

Solution Let us calculate the A.M of the given data.

$$\text{A.M } \bar{x} = \frac{12+15+18+20+25}{5}$$

$$= \frac{90}{5} = 18.$$

x	$d = x - 18$	d^2
12	-6	36
15	-3	9
$\boxed{18}$	0	0
20	2	4
25	7	49
	$\sum d = 0$	$\sum d^2 = 98$

$$\sigma = \sqrt{\frac{\sum d^2}{n}} = \sqrt{\frac{98}{5}}$$

$$= \sqrt{19.6} \simeq 4.428$$

$\therefore$ The coefficient of variation $= \dfrac{\sigma}{\bar{x}} \times 100$

$$= \frac{4.428}{18} \times 100 = \frac{442.8}{18}.$$

$\therefore$ The coefficient of variation is 24.6

Example 11.18

Following are the runs scored by two batsmen in 5 cricket matches.Who is more consistent in scoring runs.

Batsman A	38	47	34	18	33
Batsman B	37	35	41	27	35

Solution

Batsman A

x	$d = x - \bar{x}$	d^2
18	– 16	256
33	– 1	1
$\boxed{34}$	0	0
38	4	16
47	13	169
170	0	442

Now $\bar{x} = \frac{170}{5} = 34$

$$\sigma = \sqrt{\frac{\sum d^2}{n}}$$

$$= \sqrt{\frac{442}{5}} = \sqrt{88.4}$$

$$\simeq 9.4$$

Coefficient of variation, C.V $= \frac{\sigma}{\bar{x}} \times 100$

$$= \frac{9.4}{34} \times 100$$

$$= \frac{940}{34}$$

$$= 27.65$$

$\therefore$ The coefficient of variation for the runs scored by batsman A is 27.65 (1)

Batsman B

x	$d = x - \bar{x}$	d^2
27	– 8	64
35	0	0
$\boxed{35}$	0	0
37	2	4
41	6	36
175	0	104

$\bar{x} = \frac{175}{5} = 35$

$$\sigma = \sqrt{\frac{\sum d^2}{n}}$$

$$= \sqrt{\frac{104}{5}} = \sqrt{20.8}$$

$$\simeq 4.6$$

Coefficient of variation

$$= \frac{\sigma}{\bar{x}} \times 100$$

$$= \frac{4.6}{35} \times 100$$

$$= \frac{460}{35} = \frac{92}{7} = 13.14$$

$\therefore$ The coefficient of variation for the runs scored by batsman B is = 13.14 (2)

From (1) and (2), the coefficient of variation for B is less than the coefficient of variation for A.

$\therefore$ Batsman B is more consistent than the batsman A in scoring the runs.

Example 11.19

The mean of 30 items is 18 and their standard deviation is 3. Find the sum of all the items and also the sum of the squares of all the items.

Solution The mean of 30 items, $\bar{x} = 18$

The sum of 30 items, $\sum x = 30 \times 18 = 540$ $\left(\bar{x} = \frac{\sum x}{n} \right)$

Standard deviation, $\sigma = 3$

Now, $$\sigma^2 = \frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2$$

$$\Rightarrow \quad \frac{\sum x^2}{30} - 18^2 = 9$$

$$\Rightarrow \quad \frac{\sum x^2}{30} - 324 = 9$$

$$\Rightarrow \quad \sum x^2 - 9720 = 270$$

$$\sum x^2 = 9990$$

$$\therefore \quad \sum x = 540 \text{ and } \sum x^2 = 9990.$$

Example 11.20

The mean and the standard deviation of a group of 20 items was found to be 40 and 15 respectively. While checking it was found that an item 43 was wrongly written as 53. Calculate the correct mean and standard deviation.

Solution Let us find the correct mean.

$$\text{Mean of 20 items, } \bar{x} = \frac{\sum x}{n} = 40$$

$$\Rightarrow \quad \frac{\sum x}{20} = 40$$

$$\Rightarrow \quad \sum x = 20 \times 40 = 800$$

$$\text{corrected } \sum x = 800 - (\text{wrong value}) + (\text{correct value})$$

$$\text{Now,} \quad \text{corrected } \sum x = 800 - 53 + 43 = 790.$$

$$\therefore \quad \text{The corrected Mean} = \frac{790}{20} = 39.5$$

$$\text{Variance, } \sigma^2 = \frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2 = 225 \qquad \text{(given)}$$

$$\Rightarrow \quad \frac{\sum x^2}{20} - 40^2 = 225$$

$$\Rightarrow \quad \sum x^2 - 32000 = 225 \times 20 = 4500$$

$$\therefore \quad \sum x^2 = 32000 + 4500 = 36500$$

$$\text{corrected } \sum x^2 = 36500 - (\text{wrong value})^2 + (\text{correct value})^2$$

$$\text{corrected } \sum x^2 = 36500 - 53^2 + 43^2 = 36500 - 2809 + 1849$$

$$= 36500 - 960 = 35540.$$

$$\text{Now, the corrected } \sigma^2 = \frac{\text{Corrected} \sum x^2}{n} - (\text{Corrected mean})^2$$

$$= \frac{35540}{20} - (39.5)^2$$

$$= 1777 - 1560.25 = 216.75$$

$$\text{Corrected } \sigma = \sqrt{216.75} \simeq 14.72$$

$$\therefore \text{ The corrected Mean} = 39.5 \text{ and the corrected S.D.} \simeq 14.72$$

Example 11.21

For a collection of data, if $\sum x = 35$, $n = 5$, $\sum (x-9)^2 = 82$, then find $\sum x^2$ and $\sum (x-\overline{x})^2$.

Solution Given that $\sum x = 35$ and $n = 5$.

$$\therefore \quad \overline{x} = \frac{\sum x}{n} = \frac{35}{5} = 7.$$

Let us find $\sum x^2$

Now, $\sum (x-9)^2 = 82$

$$\Longrightarrow \quad \sum (x^2 - 18x + 81) = 82$$

$$\Longrightarrow \quad \sum x^2 - \left(18\sum x\right) + \left(81\sum 1\right) = 82$$

$$\Longrightarrow \quad \sum x^2 - 630 + 405 = 82 \qquad \because \sum x = 35 \text{ and } \sum 1 = 5$$

$$\Longrightarrow \quad \sum x^2 = 307.$$

To find $\sum (x-\overline{x})^2$, let us consider

$$\sum (x-9)^2 = 82$$

$$\Longrightarrow \quad \sum (x-7-2)^2 = 82$$

$$\Longrightarrow \quad \sum [(x-7)-2]^2 = 82$$

$$\Longrightarrow \quad \sum (x-7)^2 - 2\sum [(x-7)\times 2] + \sum 4 = 82$$

$$\Longrightarrow \quad \sum (x-\overline{x})^2 - 4\sum (x-\overline{x}) + 4\sum 1 = 82$$

$$\Longrightarrow \quad \sum (x-\overline{x})^2 - 4(0) + (4\times 5) = 82 \qquad \because \sum 1 = 5 \text{ and } \sum (x-\overline{x}) = 0$$

$$\Longrightarrow \quad \sum (x-\overline{x})^2 = 62$$

$$\therefore \quad \sum x^2 = 307 \text{ and } \sum (x-\overline{x})^2 = 62.$$

Example 11.22

The coefficient of variations of two series are 58 and 69. Their standard deviations are 21.2 and 15.6. What are their arithmetic means?

Solution We know that coefficient of variation, $C.V = \frac{\sigma}{\overline{x}} \times 100$.

$$\therefore \qquad \overline{x} = \frac{\sigma}{C.V} \times 100.$$

Mean of the first series, $\overline{x}_1 = \frac{\sigma}{C.V} \times 100$

$$= \frac{21.2}{58} \times 100 \qquad \because \quad C.V = 58 \text{ and } \sigma = 21.2$$

$$= \frac{2120}{58} = 36.6$$

Mean of the second series, $\overline{x}_2 = \frac{\sigma}{C.V} \times 100$

$= \frac{15.6}{69} \times 100 \qquad \because \quad C.V = 69 \text{ and } \sigma = 15.6$

$= \frac{1560}{69}$

$= 22.6$

A.M of the I series = 36.6 and the A.M of the II series = 22.6

Exercise 11.1

1. Find the range and coefficient of range of the following data.

   (i) 59, 46, 30, 23, 27, 40, 52,35, 29

   (ii) 41.2, 33.7, 29.1, 34.5, 25.7, 24.8, 56.5, 12.5

2. The smallest value of a collection of data is 12 and the range is 59. Find the largest value of the collection of data.

3. The largest of 50 measurements is 3.84kg. If the range is 0.46kg, find the smallest measurement.

4. The standard deviation of 20 observations is $\sqrt{5}$. If each observation is multiplied by 2, find the standard deviation and variance of the resulting observations.

5. Calculate the standard deviation of the first 13 natural numbers.

6. Calculate the standard deviation of the following data.

   (i) 10, 20, 15, 8, 3, 4 (ii) 38, 40, 34 ,31, 28, 26, 34

7. Calculate the standard deviation of the following data.

x	3	8	13	18	23
f	7	10	15	10	8

8. The number of books bought at a book fair by 200 students from a school are given in the following table.

No. of books	0	1	2	3	4
No of students	35	64	68	18	15

   Calculate the standard deviation.

9. Calculate the variance of the following data

x	2	4	6	8	10	12	14	16
f	4	4	5	15	8	5	4	5

10. The time (in seconds) taken by a group of people to walk across a pedestrian crossing is given in the table below.

Time (in sec.)	5-10	10-15	15-20	20-25	25-30
No. of people	4	8	15	12	11

Calculate the variance and standard deviation of the data.

11. A group of 45 house owners contributed money towards green environment of their street. The amount of money collected is shown in the table below.

Amount (₹)	0-20	20-40	40-60	60-80	80-100
No. of house owners	2	7	12	19	5

Calculate the variance and standard deviation.

12. Find the variance of the following distribution

Class interval	20-24	25-29	30-34	35-39	40-44	45-49
Frequency	15	25	28	12	12	8

13. Mean of 100 items is 48 and their standard deviation is 10. Find the sum of all the items and the sum of the squares of all the items.

14. The mean and standard deviation of 20 items are found to be 10 and 2 respectively. At the time of checking it was found that an item 12 was wrongly entered as 8. Calculate the correct mean and standard deviation.

15. If $n = 10$, $\overline{x} = 12$ and $\sum x^2 = 1530$, then calculate the coefficient of variation .

16. Calculate the coefficient of variation of the following data: 20, 18, 32, 24, 26.

17. If the coefficient of variation of a collection of data is 57 and its S.D is 6.84, then find the mean.

18. A group of 100 candidates have their average height 163.8 cm with coefficient of variation 3.2. What is the standard deviation of their heights?

19. Given $\sum x = 99$, $n = 9$ and $\sum (x - 10)^2 = 79$. Find $\sum x^2$ and $\sum (x - \overline{x})^2$.

20. The marks scored by two students A, B in a class are given below.

A	58	51	60	65	66
B	56	87	88	46	43

Who is more consistent?

Exercise 11.2

Choose the correct answer.

1. The range of the first 10 prime numbers 2, 3, 5, 7, 11, 13, 17, 19, 23 , 29 is

   (A) 28 (B) 26 (C) 29 (D) 27

2. The least value in a collection of data is 14.1. If the range of the collection is 28.4, then the greatest value of the collection is

   (A) 42.5 (B) 43.5 (C) 42.4 (D) 42.1

3. The greatest value of a collection of data is 72 and the least value is 28. Then the coefficient of range is

   (A) 44 (B) 0.72 (C) 0.44 (D) 0.28

4 For a collection of 11 items, $\Sigma x = 132$, then the arithmetic mean is

   (A) 11 (B) 12 (C) 14 (D) 13

5. For any collection of n items, $\Sigma(x - \overline{x}) =$

   (A) Σx (B) $\overline{x}$ (C) $n\overline{x}$ (D) 0

6. For any collection of n items, $(\Sigma x) - \overline{x} =$

   (A) $n\overline{x}$ (B) $(n-2)\overline{x}$ (C) $(n-1)\overline{x}$ (D) 0

7. If t is the standard deviation of $x, y. z$, then the standard deviation of $x+5,\ y+5,\ z+5$ is

   (A) $\frac{t}{3}$ (B) $t+5$ (C) t (D) $x\ y\ z$

8. If the standard deviation of a set of data is 1.6, then the variance is

   (A) 0.4 (B) 2.56 (C) 1.96 (D) 0.04

9. If the variance of a data is 12.25, then the S.D is

   (A) 3.5 (B) 3 (C) 2.5 (D) 3.25

10. Variance of the first 11 natural numbers is

   (A) $\sqrt{5}$ (B) $\sqrt{10}$ (C) $5\sqrt{2}$ (D) 10

11. The variance of 10, 10, 10, 10, 10 is

   (A) 10 (B) $\sqrt{10}$ (C) 5 (D) 0

12. If the variance of 14, 18, 22, 26, 30 is 32, then the variance of 28, 36,44,52,60 is

   (A) 64 (B) 128 (C) $32\sqrt{2}$ (D) 32

13. Standard deviation of a collection of data is $2\sqrt{2}$. If each value is multiplied by 3, then the standard deviation of the new data is

(A) $\sqrt{12}$ (B) $4\sqrt{2}$ (C) $6\sqrt{2}$ (D) $9\sqrt{2}$

14. Given $\sum(x-\overline{x})^2 = 48$, $\overline{x} = 20$ and $n = 12$. The coefficient of variation is

(A) 25 (B) 20 (C) 30 (D) 10

15. Mean and standard deviation of a data are 48 and 12 respectively. The coefficient of variation is

(A) 42 (B) 25 (C) 28 (D) 48

Points to Remember

- (i) Range $= L - S$, the difference between the greatest and the least of the observations.

  (ii) Coefficient of range $= \dfrac{L-S}{L+S}$.

- Standard deviation for an ungrouped data

  (i) $\sigma = \sqrt{\dfrac{\sum d^2}{n}}$, where $d = x - \overline{x}$ and $\overline{x}$ is the mean.

  (ii) $\sigma = \sqrt{\dfrac{\sum d^2}{n} - \left(\dfrac{\sum d}{n}\right)^2}$, where $d = x - A$ and A is the assumed mean.

- Standard deviation for a grouped data

  (i) $\sigma = \sqrt{\dfrac{\sum fd^2}{\sum f}}$, where $d = x - \overline{x}$ and $\overline{x}$ is the mean.

  (ii) $\sigma = \sqrt{\dfrac{\sum fd^2}{\sum f} - \left(\dfrac{\sum fd}{\sum f}\right)^2}$, where $d = x - A$ and A is the assumed mean.

- Standard deviation of a collection of data remains unchanged when each value is added or subtracted by a constant.

- Standard deviation of a collection of data gets multiplied or divided by the quantity k, if each item is multiplied or divided by k.

- Standard deviation of the first n natural numbers, $\sigma = \sqrt{\dfrac{n^2-1}{12}}$.

- Variance is the square of standard deviation.

- Coefficient of variation, C.V. $= \dfrac{\sigma}{\overline{x}} \times 100$. It is used for comparing the consistency of two or more collections of data.

12

PROBABILITY

It is remarkable that a science which began with the consideration of games of chance should have become the most important object of human knowledge -P.D. Laplace.

- Introduction
- Classical Definition
- Addition Theorem

Pierre de Laplace

(1749-1827)
France

Laplace remembered as one of the greatest scientists of all time, sometimes referred to as a French Newton.

In 1812, Laplace established many fundamental results in statistics. He put forth a mathematical system of inductive reasoning based on probability. He only introduced the principles of probability, one among them is "probability is the ratio of the favoured events to the total possible events".

12.1 Introduction

In every day life, almost everything that we see or do is subject to chance. The occurrences of events like Earthquakes, Cyclones, Tsunami, Lightning, Epidemics, etc... are unpredictable. Most of the events occur quite unexpectedly and result in heavy loss to humanity. If we predict the occurrences of such events well in advance based on their past occurrences with a good amount of accuracy, one can think of preventive measures or damage control exercises much to the relief of human society. Such predictions well in advance of their actual happenings require the study of Probability theory.

A gambler's dispute-problem posed by Chevalier de Mere in 1654 led to exchange of letters between two famous French Mathematicians Blasie Pascal and Pierre de Fermat which created a mathematical theory of Probability. The family of major contributors to the development of Probability theory includes mathematicians like Christian Huggens (1629-1695), Bernoulli (1654-1705), De-Moivre (1667-1754), Pierre de Laplace (1749-1827), Gauss (1777-1855), Poisson (1781-1845), Chebyshev (1821-1894), Markov (1856-1922). In 1933, a Russian Mathematician A. Kolmogorov introduced an axiomatic approach which is considered as the basis for Modern Probability theory.

Probabilities always pertain to the occurrence or nonoccurrence of events. Let us define the terms random experiment, trial, sample space and different types of events used in the study of probability.

Mathematicians use the words "experiment" and "outcome" in a very wide sense. Any process of observation or measurement is called an experiment. Noting down whether a newborn baby is male or female, tossing a coin, picking up a ball from a bag containing balls of different colours and observing the number of accidents at a particular place in a day are some examples of experiments.

A random experiment is one in which the exact outcome cannot be predicted before conducting the experiment. However, one can list out all possible outcomes of the experiment.

The set of all possible outcomes of a random experiment is called its sample space and it is denoted by the letter *S*. Each repetition of the experiment is called a trial.

A subset of the sample space *S* is called an event.

Let *A* be a subset of *S*. If the experiment, when conducted, results in an outcome that belongs to *A*, then we say that the event *A* has occurred.

Let us illustrate random experiment, sample space, events with the help of some examples.

Random Experiment	Sample Space	Some Events
Tossing an unbiased coin once	$S = \{H, T\}$	The occurrence of head, $\{H\}$ is an event. The occurrence of tail, $\{T\}$ is another event.
Tossing an unbiased coin twice	$S = \{HT, HH, TT, TH\}$	$\{HT, HH\}$ and $\{TT\}$ are some of the events
Rolling an unbiased die once	$S = \{1, 2, 3, 4, 5, 6\}$	$\{1,3,5\}, \{2,4,6\}, \{3\}$ and $\{6\}$ are some of the events

Equally likely events

Two or more events are said to be equally likely if each one of them has an equal chance of occurrence.

In tossing a coin, the occurrence of Head and the occurrence of Tail are equally likely events.

Mutually exclusive events

Two or more events are said to be mutually exclusive if the occurrence of one event prevents the occurrence of other events. That is, mutually exclusive events can't occur simultaneously. Thus, if *A* and *B* are two mutually exclusive events, then $A \cap B = \phi$.

Fig. 12.1

In tossing a coin, the occurrence of head excludes the occurrence of tail. Similarly if an unbiased die is rolled, the six possible outcomes are mutually exclusive, since two or more faces cannot turn up simultaneously.

Complementary events

Let E be an event of a random experiment and S be its sample space. The set containing all the other outcomes which are not in E but in the sample space is called the complimentary event of E. It is denoted by $\overline{E}$. Thus, $\overline{E} = S - E$. Note that E and $\overline{E}$ are mutually exclusive events.

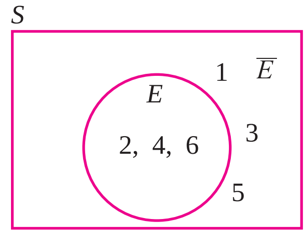


Fig. 12.2

In throwing a die, let $E = \{2, 4, 6\}$ be an event of getting a multiple of 2.

Then the complementary of the event E is given by $\overline{E} = \{1, 3, 5\}$.(see Figure 12.2)

Exhaustive events

Events $E_1, E_2, \cdots, E_n$ are exhaustive events if their union is the sample space S.

Sure event

The sample space of a random experiment is called sure or certain event as any one of its elements will surely occur in any trail of the experiment.

For example, getting one of 1, 2, 3, 4, 5, 6 in rolling a die is a sure event.

Impossible event

An event which will not occur on any account is called an impossible event.

It is denoted by ϕ.

For example, getting 7 in rolling a die once is an impossible event.

Favourable outcomes

The outcomes corresponding to the occurrence of the desired event are called favourable outcomes of the event.

For example, if E is an event of getting an odd number in rolling a die, then the outcomes 1, 3, 5 are favourable to the event E.

Note

In this chapter, we consider only random experiments all of whose outcomes are equally likely and sample spaces are finite. Thus, whenever we refer coins or dice, they are assumed to be unbiased.

12.2 Classical definition of probability

If a sample space contains n outcomes and if m of them are favourable to an event A, then, we write $n(S) = n$ and $n(A) = m$. The Probability of the event A, denoted by $P(A)$, is defined as the ratio of m to n.

That is $P(A) = \dfrac{\text{number of outcomes favourable to } A}{\text{total number of outcomes}}.$

$$\therefore\ P(A) = \frac{n(A)}{n(S)} = \frac{m}{n}.$$

Note

(i) The above classical definition of probability is not applicable if the number of possible outcomes is infinite and the outcomes are not equally likely.

(ii) The probability of an event A lies between 0 and 1, both inclusive;

That is $0 \le P(A) \le 1$.

(iii) The probability of the sure event is 1. That is $P(S) = 1$.

(iv) The probability of an impossible event is 0. That is $P(\phi) = 0$.

(v) The probability that the event A will not occur is given by

$$P(\text{not } A) = P(\overline{A}) \text{ or } P(A') = \frac{n-m}{n} = \frac{n}{n} - \frac{m}{n}$$

$$\Longrightarrow\ P(\overline{A}) = 1 - \frac{m}{n} = 1 - P(A).$$

(vi) $P(A) + P(\overline{A}) = 1$.

Example 12.1

A fair die is rolled. Find the probability of getting

(i) the number 4 (ii) an even number

(iii) a prime factor of 6 (iv) a number greater than 4.

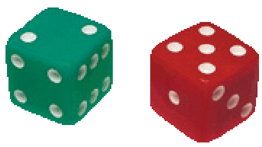


Fig. 12.3

Solution In rolling a die, the sample space $S = \{1, 2, 3, 4, 5, 6\}$.

$\therefore n(S) = 6$.

(i) Let A be the event of getting 4.

$A = \{4\}$ $\therefore n(A) = 1$.

$$\therefore P(A) = \frac{n(A)}{n(S)} = \frac{1}{6}.$$

(ii) Let B be the event of getting an even number.

$B = \{2, 4, 6\}$ $\therefore$ $n(B) = 3$.

$$\text{Hence } P(B) = \frac{n(B)}{n(S)} = \frac{3}{6} = \frac{1}{2}.$$

(iii) Let C be the event of getting a prime factor of 6.

Then $C = \{2, 3\}$ $\therefore$ $n(C) = 2$.

Hence $P(C) = \dfrac{n(C)}{n(S)} = \dfrac{2}{6} = \dfrac{1}{3}$.

(iv) Let D be the event of getting a number greater than 4.

$D = \{5, 6\}$ $n(D) = 2$.

Hence, $P(D) = \dfrac{n(D)}{n(S)} = \dfrac{2}{6} = \dfrac{1}{3}$.

Example 12.2

In tossing a fair coin twice, find the probability of getting

(i) two heads (ii) atleast one head (iii) exactly one tail

Solution In tossing a coin twice, the sample space

$$S = \{HH, HT, TH, TT\}$$

$\therefore$ $n(S) = 4$.

(i) Let A be the event of getting two heads. Then $A = \{HH\}$.

Thus, $n(A) = 1$.

$\therefore$ $P(A) = \dfrac{n(A)}{n(S)} = \dfrac{1}{4}$.

(ii) Let B be the event of getting at least one head. Then $B = \{HH, HT, TH\}$

Thus, $n(B) = 3$.

$\therefore$ $P(B) = \dfrac{n(B)}{n(S)} = \dfrac{3}{4}$.

(iii) Let C be the event of getting exactly one tail. Then $C = \{HT, TH\}$

Thus, $n(C) = 2$.

$\therefore$ $P(C) = \dfrac{n(C)}{n(S)} = \dfrac{2}{4} = \dfrac{1}{2}$.

Example 12.3

An integer is chosen from the first twenty natural numbers. What is the probability that it is a prime number?

Solution Here $S = \{1, 2, 3, \cdots, 20\}$

$\therefore$ $n(S) = 20$.

Let A be the event of choosing a prime number.

Then, $A = \{2, 3, 5, 7, 11, 13, 17, 19\}$.

$n(A) = 8$.

Hence, $P(A) = \dfrac{n(A)}{n(S)} = \dfrac{8}{20} = \dfrac{2}{5}$.

Example 12.4

There are 7 defective items in a sample of 35 items. Find the probability that an item chosen at random is non-defective.

Solution Total number of items $n(S) = 35$.

Number of defective items = 7.

Let A be the event of choosing a non-defective item.

Number of non-defective items, $n(A) = 35 - 7 = 28$.

$\therefore$ Probability that the chosen item is non-defective,

$$P(A) = \frac{n(A)}{n(S)} = \frac{28}{35} = \frac{4}{5}.$$

Example 12.5

Two unbiased dice are rolled once. Find the probability of getting

(i) a sum 8 (ii) a doublet (iii) a sum greater than 8.

Solution When two dice are thrown, the sample space is

$$S = \{(1,1),\ (1,2),\ (1,3),\ (1,4),\ (1,5),\ (1,6),$$
$$(2,1),\ (2,2),\ (2,3),\ (2,4),\ (2,5),\ (2,6),$$
$$(3,1),\ (3,2),\ (3,3),\ (3,4),\ (3,5),\ (3,6),$$
$$(4,1),\ (4,2),\ (4,3),\ (4,4),\ (4,5),\ (4,6),$$
$$(5,1),\ (5,2),\ (5,3),\ (5,4),\ (5,5),\ (5,6),$$
$$(6,1),\ (6,2),\ (6,3),\ (6,4),\ (6,5),\ (6,6)\ \}$$

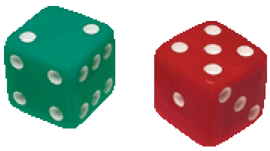


Fig. 12.4

$$\therefore\ n(S) = 6 \times 6 = 36.$$

(i) Let A be the event of getting a sum 8.

$$\therefore\quad A = \{(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)\}.$$

Then $n(A) = 5$.

Hence, $P(A) = \dfrac{n(A)}{n(S)} = \dfrac{5}{36}$.

(ii) Let B be the event of getting a doublet

$$\therefore\quad B = \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6)\}.$$

Thus, $n(B) = 6$.

$\therefore$ $P(B) = \dfrac{n(B)}{n(S)} = \dfrac{6}{36} = \dfrac{1}{6}$.

(iii) Let C be the event of getting a sum greater than 8.

Then, $C = \{(3,6),(4,5),(4,6),(5,4),(5,5),(5,6),(6,3),(6,4),(6,5),(6,6)\}$.

Thus, $n(C) = 10$.

$\therefore$ $P(C) = \dfrac{n(C)}{n(S)} = \dfrac{10}{36} = \dfrac{5}{18}$.

Example 12.6

From a well shuffled pack of 52 playing cards, one card is drawn at random. Find the probability of getting

(i) a king (ii) a black king

(iii) a spade card (iv) a diamond 10.

The 52 playing cards are classified as

Spade ♠	Hearts ♥	Clavor ♣	Diamond ♦
A	A	A	A
2	2	2	2
3	3	3	3
4	4	4	4
5	5	5	5
6	6	6	6
7	7	7	7
8	8	8	8
9	9	9	9
10	10	10	10
J	J	J	J
Q	Q	Q	Q
K	K	K	K
13	13	13	13

Solution Now, $n(S) = 52$.

(i) Let A be the event of drawing a king card

$$\therefore n(A) = 4.$$

$$\therefore P(A) = \frac{n(A)}{n(S)} = \frac{4}{52} = \frac{1}{13}.$$

(ii) Let B be the event of drawing a black king card

Thus, $n(B) = 2$.

$$\therefore P(B) = \frac{n(B)}{n(S)} = \frac{2}{52} = \frac{1}{26}.$$

(iii) Let C be the event of drawing a spade card

Thus, $n(C) = 13$.

$$\therefore P(C) = \frac{n(C)}{n(S)} = \frac{13}{52} = \frac{1}{4}.$$

(iv) Let D be the event of drawing a diamond 10 card.

Thus, $n(D) = 1$.

$$P(D) = \frac{n(D)}{n(S)} = \frac{1}{52}.$$

Example 12.7

There are 20 boys and 15 girls in a class of 35 students . A student is chosen at random. Find the probability that the chosen student is a (i) boy (ii) girl.

Solution Let S be the sample space of the experiment.

Let B and G be the events of selecting a boy and a girl respectively.

$$\therefore\ n(S) = 35,\ n(B) = 20 \text{ and } n(G) = 15.$$

(i) Probability of choosing a boy is $P(B) = \frac{n(B)}{n(S)} = \frac{20}{35}$

$$\Longrightarrow P(B) = \frac{4}{7}.$$

(ii) Probability of choosing a girl is $P(G) = \frac{n(G)}{n(S)} = \frac{15}{35}$

$$\Longrightarrow P(G) = \frac{3}{7}.$$

Example 12.8

The probability that it will rain on a particular day is 0.76. What is the probability that it will not rain on that day?

Solution Let A be the event that it will rain. Then $\overline{A}$ is the event that it will not rain.

Given that $P(A) = 0.76.$

Thus, $P(\overline{A}) = 1 - 0.76 \qquad \because P(A) + P(\overline{A}) = 1$

$= 0.24.$

$\therefore$ The probability that it will not rain is 0.24.

Example 12.9

A bag contains 5 red balls and some blue balls. If the probability of drawing a blue ball from the bag is thrice that of drawing a red ball, then find the number of blue balls in the bag.

Solution Let the number of blue balls be x.

$\therefore$ Total number of balls, $n(S) = 5 + x$.

Let B be the event of drawing a blue ball and R be the event of drawing a red ball.

Given $P(B) = 3P(R)$

$$\Longrightarrow \quad \frac{n(B)}{n(S)} = 3\frac{n(R)}{n(S)}$$

$$\Longrightarrow \quad \frac{x}{5+x} = 3\left(\frac{5}{5+x}\right)$$

$$\Longrightarrow \quad x = 15$$

Thus, number of blue balls $= 15$.

Example 12.10

Find the probability that

(i) a leap year selected at random will have 53 Fridays

(ii) a leap year selected at random will have only 52 Fridays

(iii) a non-leap year selected at random will have 53 Fridays.

Solution (i) Number of days in a leap year = 366 days. i.e., 52 weeks and 2 days.

Now 52 weeks contain 52 Fridays and the remaining two days will be one of the following seven possibilities.

(Sun, Mon), (Mon, Tue), (Tue, Wed), (Wed, Thur), (Thur, Fri), (Fri, Sat) and (Sat, Sun).

The probability of getting 53 Fridays in a leap year is same as the probability of getting a Friday in the above seven possibilities.

Here $S = \{(\text{Sun},\text{Mon}),(\text{Mon},\text{Tue}),(\text{Tue},\text{Wed}),(\text{Wed},\text{Thur}),(\text{Thur},\text{Fri}),(\text{Fri},\text{Sat}),(\text{Sat},\text{Sun})\}$.

Then $n(S) = 7$.

Let A be the event of getting one Friday in the remaining two days.

$$A = \{(Thur,Fri),(Fri,Sat)\} \qquad \text{Then } n(A) = 2.$$

$$p(A) = \frac{n(A)}{n(S)} = \frac{2}{7}.$$

(ii) To get only 52 Fridays in a leap year, there must be no Friday in the remaining two days. Let B be the event of not getting a Friday in the remaining two days. Then

$$B = \{(\text{Sun},\text{Mon}),(\text{Mon},\text{Tue}),(\text{Tue},\text{Wed}),(\text{Wed},\text{Thur}),(\text{Sat},\text{Sun})\}.$$

$$n(B) = 5.$$

Now, $P(B) = \frac{n(B)}{n(S)} = \frac{5}{7}$.

Note that A and B are complementary events.

(iii) Number of days in a non leap year = 365 days. i.e., 52 weeks and 1 day.

To get 53 Fridays in a non leap year, there must be a Friday in the seven possibilities: Sun, Mon, Tue, Wed, Thur, Fri and Sat.

Here $S = \{\text{Sun, Mon, Tue, Wed, Thur, Fri and Sat}\}$.

$\therefore \quad n(S) = 7$.

Let C be the event of getting a Friday in the remaining one day . Then

$$C = \{\text{Fri}\} \implies n(C) = 1.$$

$$\therefore P(C) = \frac{n(C)}{n(S)} = \frac{1}{7}.$$

Example 12.11

If A is an event of a random experiment such that $P(A):P(\overline{A}) = 7:12$, then find $P(A)$.

Solution Given that $P(A):P(\overline{A}) = 7:12$.

Let $P(A) = 7k$ and $P(\overline{A}) = 12k$., $k > 0$

We know that $P(A) + P(\overline{A}) = 1$.

Then, $7k + 12k = 1 \implies 19k = 1$.

Thus, $k = \frac{1}{19}$

$\therefore \quad P(A) = 7k = \frac{7}{19}$.

Aliter

$$\frac{P(A)}{P(\overline{A})} = \frac{7}{12}$$

$$12P(A) = 7 \times P(\overline{A})$$

$$= 7\,[1-P(A)]$$

$$19P(A) = 7$$

Thus, $P(A) = \frac{7}{19}$

Exercise 12. 1

1. A ticket is drawn from a bag containing 100 tickets. The tickets are numbered from one to hundred. What is the probability of getting a ticket with a number divisible by 10?
2. A die is thrown twice. Find the probability of getting a total of 9.
3. Two dice are thrown together. Find the probability that the two digit number formed with the two numbers turning up is divisible by 3.
4. Three rotten eggs are mixed with 12 good ones. One egg is chosen at random. What is the probability of choosing a rotten egg?
5. Two coins are tossed together. What is the probability of getting at most one head.
6. One card is drawn randomly from a well shuffled deck of 52 playing cards. Find the probability that the drawn card is

   (i) a Diamond (ii) not a Diamond (iii) not an Ace.
7. Three coins are tossed simultaneously. Find the probability of getting

   (i) at least one head (ii) exactly two tails (iii) at least two heads.
8. A bag contains 6 white balls numbered from 1 to 6 and 4 red balls numbered from 7 to 10. A ball is drawn at random. Find the probability of getting

   (i) an even-numbered ball (ii) a white ball.
9. A number is selected at random from integers 1 to 100. Find the probability that it is

   (i) a perfect square (ii) not a perfect cube.
10. For a sightseeing trip, a tourist selects a country randomly from Argentina, Bangladesh, China, Angola, Russia and Algeria. What is the probability that the name of the selected country will begin with *A* ?
11. A box contains 4 Green, 5 Blue and 3 Red balls. A ball is drawn at random. Find the probability that the selected ball is (i) Red in colour (ii) not Green in colour.
12. 20 cards are numbered from 1 to 20. One card is drawn at random. What is the probability that the number on the card is

    (i) a multiple of 4 (ii) not a multiple of 6.
13. A two digit number is formed with the digits 3, 5 and 7. Find the probability that the number so formed is greater than 57 (repetition of digits is not allowed).
14. Three dice are thrown simultaneously. Find the probability of getting the same number on all the three dice.

15. Two dice are rolled and the product of the outcomes (numbers) are found. What is the probability that the product so found is a prime number?
16. A jar contains 54 marbles each of which is in one of the colours blue, green and white. The probability of drawing a blue marble is $\frac{1}{3}$ and the probability of drawing a green marble is $\frac{4}{9}$. How many white marbles does the jar contain?
17. A bag consists of 100 shirts of which 88 are good, 8 have minor defects and 4 have major defects. A trader A will accept only the shirt which are good, but the trader B will not accept the shirts which have major defects. One shirt is drawn at random. What is the probability that it is acceptable by (i) A (ii) B ?
18. A bag contains 12 balls out of which x balls are white. (i) If one ball is drawn at random, what is the probability that it will be a white ball. (ii) If 6 more white balls are put in the bag and if the probability of drawing a white ball will be twice that of in (i), then find x.
19. Piggy bank contains 100 fifty-paise coins, 50 one-rupee coins, 20 two-rupees coins and 10 five- rupees coins. One coin is drawn at random. Find the probability that the drawn coin (i) will be a fifty-paise coin (ii) will not be a five-rupees coin.

12.3 Addition theorem on probability

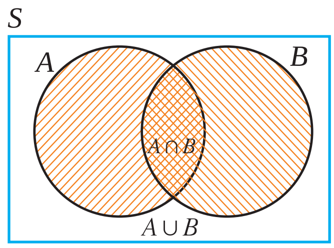


Fig. 12.5

Let A and B be subsets of a finite non-empty set S. Then

$$n(A \cup B) = n(A) + n(B) - n(A \cap B).$$

Divide both sides by $n(S)$,we get

$$\frac{n(A \cup B)}{n(S)} = \frac{n(A)}{n(S)} + \frac{n(B)}{n(S)} - \frac{n(A \cap B)}{n(S)} \qquad (1)$$

If the subsets A and B correspond to two events A and B of a random experiment and if the set S corresponds to the sample space S of the experiment, then (1) becomes

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

This result is known as the addition theorem on probability.

Note

(i) The event $A \cup B$ is said to occur if the event A occurs or the event B occurs or both A and B occur simultaneously. The event $A \cap B$ is said to occur if both the events A and B occur simultaneously.

(ii) If A and B are mutually exclusive events, then $A \cap B = \emptyset$.

Thus, $P(A \cup B) = P(A) + P(B) \qquad \because P(A \cap B) = 0$.

(iii) $A \cap \overline{B}$ is same as $A \setminus B$ in the language of set theory.

Results (without proof)

(i) If A, B and C are any 3 events associated with a sample space S, then

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C).$$

(ii) If A_1, A_2 and A_3 are three mutually exclusive events, then

$$P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3).$$

(iii) If $A_1, A_2, A_3, \cdots, A_n$ are mutually exclusive events, then

$$P(A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n) = P(A_1) + P(A_2) + P(A_3) + \cdots + P(A_n).$$

(iv) $P(A \cap \overline{B}) = P(A) - P(A \cap B)$,

$P(\overline{A} \cap B) = P(B) - P(A \cap B)$

where $A \cap \overline{B}$ mean only A and not B;

Similarly $\overline{A} \cap B$ means only B and not A.

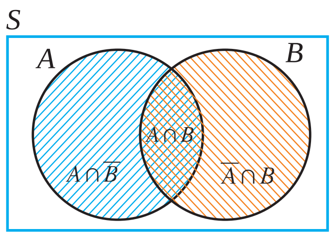


Fig. 12.6

Example 12.12

Three coins are tossed simultaneously. Using addition theorem on probability, find the probability that either exactly two tails or at least one head turn up.

Solution Now the sample space $S = \{HHH, HHT, HTH, HTT, TTT, TTH, THT, THH\}$.

Hence, $n(S) = 8$.

Let A be the event of getting exactly two tails.

Thus, $A = \{HTT, TTH, THT\}$ and hence $n(A) = 3$.

$$\therefore \quad P(A) = \frac{n(A)}{n(S)} = \frac{3}{8}.$$

Let B be the event of getting at least one head.

Thus, $B = \{HTT, THT, TTH, HHT, HTH, THH, HHH\}$ and hence $n(B) = 7$.

$$\therefore \quad P(B) = \frac{n(B)}{n(S)} = \frac{7}{8}.$$

Now, the events A and B are not mutually exclusive.

Since $A \cap B = A$, $P(A \cap B) = P(A) = \frac{3}{8}$.

$\therefore$ $P(A \text{ or } B) = P(A) + P(B) - P(A \cap B)$

Thus $P(A \cup B) = \frac{3}{8} + \frac{7}{8} - \frac{3}{8} = \frac{7}{8}$.

Note

In the above problem, we applied addition theorem on probability.

However, one can notice that $A \cup B = B$. Thus, $P(A \cup B) = P(B) = \frac{7}{8}$.

Example 12.13

A die is thrown twice. Find the probability that at least one of the two throws comes up with the number 5 (use addition theorem).

Solution In rolling a die twice, the size of the sample space , $n(S) = 36$.

Let A be the event of getting 5 in the first throw.

$\therefore \quad A = \{(5,1),(5,2),(5,3),(5,4),(5,5),(5,6)\}$.

Thus, $\quad n(A) = 6$, and $\quad P(A) = \frac{6}{36}$.

Let B be the event of getting 5 in the second throw.

$\therefore \quad B = \{(1,5),(2,5),(3,5),(4,5),(5,5),(6,5)\}$.

Thus, $\quad n(B) = 6$ and $\quad P(B) = \frac{6}{36}$.

A and B are not mutually exclusive events , since $\quad A \cap B = \{(5,5)\}$.

$\therefore \quad n(A \cap B) = 1$ and $\quad P(A \cap B) = \frac{1}{36}$.

$\therefore$ By addition theorem,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$
$$= \frac{6}{36} + \frac{6}{36} - \frac{1}{36} = \frac{11}{36}.$$

Example 12.14

The probability that a girl will be selected for admission in a medical college is 0.16. The probability that she will be selected for admission in an engineering college is 0.24 and the probability that she will be selected in both, is 0.11

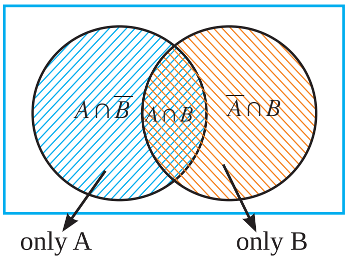


Fig. 12.7

(i) Find the probability that she will be selected in at least one of the two colleges.

(ii) Find the probability that she will be selected either in a medical college only or in an engineering college only.

Solution Let A be the event of getting selected in a medical college and B be the event of getting selected for admission in an engineering college.

(i) $\quad P(A) = 0.16,\ P(B) = 0.24$ and $P(A \cap B) = 0.11$

P (she will be selected for admission in at least one of the two colleges) is

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= 0.16 + 0.24 - 0.11 = 0.29$$

(ii) P(she will be selected for admission in only one of the two colleges)

$= P$ (only A or only B)

$= P(A \cap \overline{B}) + P(\overline{A} \cap B)$

$= [P(A) - P(A \cap B)] + [P(B) - P(A \cap B)]$

$= (0.16 - 0.11) + (0.24 - 0.11) = 0.18.$

Example 12.15

A letter is chosen at random from the letters of the word "ENTERTAINMENT". Find the probability that the chosen letter is a vowel or T. (repetition of letters is allowed)

Solution There are 13 letters in the word ENTERTAINMENT.

$\therefore \quad n(S) = 13.$

Let A be the event of getting a vowel.

$\therefore \quad n(A) = 5.$

Hence, $\quad P(A) = \dfrac{n(A)}{n(S)} = \dfrac{5}{13}.$

Let B be the event of getting the letter T.

$\therefore \quad n(B) = 3$

Hence, $\quad P(B) = \dfrac{n(B)}{n(S)} = \dfrac{3}{13}$. Then

$P(A \text{ or } B) = P(A) + P(B)$ $\quad \because$ A and B are mutually exclusive events

$= \dfrac{5}{13} + \dfrac{3}{13} = \dfrac{8}{13}.$

Example 12.16

Let A, B, C be any three mutually exclusive and exhaustive events such that

$P(B) = \frac{3}{2}P(A)$ and $P(C) = \frac{1}{2}P(B)$. Find $P(A)$.

Solution

Let $P(A) = p$.

Now, $P(B) = \frac{3}{2}P(A) = \frac{3}{2}p$.

Also, $P(C) = \frac{1}{2}P(B) = \frac{1}{2}\left(\frac{3}{2}p\right) = \frac{3}{4}p$.

Given that A, B and C are mutually exclusive and exhaustive events.

$\therefore \; P(A \cup B \cup C) = P(A) + P(B) + P(C)$ and $S = A \cup B \cup C$.

Now, $\quad P(S) = 1.$

That is, $P(A) + P(B) + P(C) = 1$

$$\Longrightarrow \quad p + \frac{3}{2}p + \frac{3}{4}p = 1$$

$$\Longrightarrow \quad 4p + 6p + 3p = 4$$

Thus, $p = \dfrac{4}{13}$

Hence, $P(A) = \dfrac{4}{13}$.

Example 12.17

A card is drawn from a deck of 52 cards. Find the probability of getting a King or a Heart or a Red card.

Solution Let A, B and C be the events of getting a King, a Heart and a Red card respectively.

Now, $n(S) = 52$, $n(A) = 4$, $n(B) = 13$, $n(C) = 26$. Also,

$n(A \cap B) = 1$, $n(B \cap C) = 13$, $n(C \cap A) = 2$ and $n(A \cap B \cap C) = 1$.

$$\therefore \; P(A) = \frac{4}{52},\; P(B) = \frac{13}{52},\; P(C) = \frac{26}{52}.$$

$$P(A \cap B) = \frac{1}{52},\; P(B \cap C) = \frac{13}{52},\; P(C \cap A) = \frac{2}{52} \text{ and } P(A \cap B \cap C) = \frac{1}{52}.$$

Now $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$

$$= \frac{4}{52} + \frac{13}{52} + \frac{26}{52} - \frac{1}{52} - \frac{13}{52} - \frac{2}{52} + \frac{1}{52} = \frac{44-16}{52}$$

$$= \frac{7}{13}.$$

Example 12.18

A bag contains 10 white, 5 black, 3 green and 2 red balls. One ball is drawn at random. Find the probability that the ball drawn is white or black or green.

Solution Let S be the sample space.

$$\therefore \quad n(S) = 20.$$

Let W, B and G be the events of selecting a white, black and green ball respectively.

Probability of getting a white ball, $P(W) = \dfrac{n(W)}{n(S)} = \dfrac{10}{20}$.

Probability of getting a black ball, $P(B) = \dfrac{n(B)}{n(S)} = \dfrac{5}{20}$.

Probability of getting a green ball, $P(G) = \dfrac{n(G)}{n(S)} = \dfrac{3}{20}$.

$\therefore$ Probability of getting a white or black or green ball,

$P(W \cup B \cup G) = P(W) + P(B) + P(G)$ $\quad \because$ W, B and G are mutually exclusive.

$$= \frac{10}{20} + \frac{5}{20} + \frac{3}{20} = \frac{9}{10}.$$

(Note : $P(W \cup B \cup G) = P(R') = 1 - P(R) = 1 - \dfrac{2}{20} = \dfrac{9}{10}$)

Exercise 12.2

1. If A and B are mutually exclusive events such that $P(A)=\frac{3}{5}$ and $P(B)=\frac{1}{5}$, then find $P(A\cup B)$.
2. If A and B are two events such that $P(A)=\frac{1}{4}, P(B)=\frac{2}{5}$ and $P(A\cup B)=\frac{1}{2}$,then find $P(A\cap B)$.
3. If $P(A)=\frac{1}{2}, P(B)=\frac{7}{10}, P(A\cup B)=1$. Find (i) $P(A\cap B)$ (ii) $P(A'\cup B')$.
4. If a die is rolled twice, find the probability of getting an even number in the first time or a total of 8 .
5. One number is chosen randomly from the integers 1 to 50. Find the probability that it is divisible by 4 or 6.
6. A bag contains 50 bolts and 150 nuts. Half of the bolts and half of the nuts are rusted. If an item is chosen at random, find the probability that it is rusted or that it is a bolt.
7. Two dice are rolled simultaneously. Find the probability that the sum of the numbers on the faces is neither divisible by 3 nor by 4.
8. A basket contains 20 apples and 10 oranges out of which 5 apples and 3 oranges are rotten. If a person takes out one fruit at random, find the probability that the fruit is either an apple or a good fruit.
9. In a class, 40% of the students participated in Mathematics-quiz, 30% in Science-quiz and 10% in both the quiz programmes. If a student is selected at random from the class, find the probability that the student participated in Mathematics or Science or both quiz programmes.
10. A card is drawn at random from a well-shuffled deck of 52 cards. Find the probability that it will be a spade or a king.
11. A box contains 10 white, 6 red and 10 black balls. A ball is drawn at random. Find the probability that the ball drawn is white or red.
12. A two digit number is formed with the digits 2, 5, 9 (repetition is allowed). Find the probability that the number is divisible by 2 or 5.
13. Each individual letter of the word “ACCOMMODATION” is written in a piece of paper, and all 13 pieces of papers are placed in a jar. If one piece of paper is selected at random from the jar, find the probability that
    (i) the letter ‘A’ or ‘O’ is selected.
    (ii) the letter ‘M’ or ‘C’ is selected.

14. The probability that a new car will get an award for its design is 0.25, the probability that it will get an award for efficient use of fuel is 0.35 and the probability that it will get both the awards is 0.15. Find the probability that

    (i) it will get atleast one of the two awards

    (ii) it will get only one of the awards.

15. The probability that A, B and C can solve a problem are $\frac{4}{5}, \frac{2}{3}$ and $\frac{3}{7}$ respectively. The probability of the problem being solved by A and B is $\frac{8}{15}$, B and C is $\frac{2}{7}$, A and C is $\frac{12}{35}$. The probability of the problem being solved by all the three is $\frac{8}{35}$. Find the probability that the problem can be solved by atleast one of them.

Exercise 12.3

Choose the correct answer

1. If ϕ is an impossible event, then $P(\phi) =$

   (A) 1 (B) $\frac{1}{4}$ (C) 0 (D) $\frac{1}{2}$

2. If S is the sample space of a random experiment, then $P(S) =$

   (A) 0 (B) $\frac{1}{8}$ (C) $\frac{1}{2}$ (D) 1

3. If p is the probability of an event A, then p satisfies

   (A) $0 < p < 1$ (B) $0 \le p \le 1$ (C) $0 \le p < 1$ (D) $0 < p \le 1$

4. Let A and B be any two events and S be the corresponding sample space. Then $P(\overline{A} \cap B) =$

   (A) $P(B) - P(A \cap B)$ (B) $P(A \cap B) - P(B)$

   (C) $P(S)$ (D) $P[(A \cup B)']$

5. The probability that a student will score centum in mathematics is $\frac{4}{5}$. The probability that he will not score centum is

   (A) $\frac{1}{5}$ (B) $\frac{2}{5}$ (C) $\frac{3}{5}$ (D) $\frac{4}{5}$

6. If A and B are two events such that $P(A) = 0.25, P(B) = 0.05$ and $P(A \cap B) = 0.14$, then $P(A \cup B) =$

   (A) 0.61 (B) 0.16 (C) 0.14 (D) 0.6

7. There are 6 defective items in a sample of 20 items. One item is drawn at random. The probability that it is a non-defective item is

   (A) $\frac{7}{10}$ (B) 0 (C) $\frac{3}{10}$ (D) $\frac{2}{3}$

8. If A and B are mutually exclusive events and S is the sample space such that $P(A) = \frac{1}{3}P(B)$ and $S = A \cup B$, then $P(A) =$

(A) $\frac{1}{4}$ (B) $\frac{1}{2}$ (C) $\frac{3}{4}$ (D) $\frac{3}{8}$

9. The probabilities of three mutually exclusive events A, B and C are given by $\frac{1}{3}, \frac{1}{4}$, and $\frac{5}{12}$. Then $P(A \cup B \cup C)$ is

(A) $\frac{19}{12}$ (B) $\frac{11}{12}$ (C) $\frac{7}{12}$ (D) 1

10. If $P(A) = 0.25, P(B) = 0.50, P(A \cap B) = 0.14$ then $P(\text{neither } A \text{ nor } B) =$

(A) 0.39 (B) 0.25 (C) 0.11 (D) 0.24

11. A bag contains 5 black balls, 4 white balls and 3 red balls. If a ball is selected at random, the probability that it is not red is

(A) $\frac{5}{12}$ (B) $\frac{4}{12}$ (C) $\frac{3}{12}$ (D) $\frac{3}{4}$

12. Two dice are thrown simultaneously. The probability of getting a doublet is

(A) $\frac{1}{36}$ (B) $\frac{1}{3}$ (C) $\frac{1}{6}$ (D) $\frac{2}{3}$

13. A fair die is thrown once. The probability of getting a prime or composite number is

(A) 1 (B) 0 (C) $\frac{5}{6}$ (D) $\frac{1}{6}$

14. Probability of getting 3 heads or 3 tails in tossing a coin 3 times is

(A) $\frac{1}{8}$ (B) $\frac{1}{4}$ (C) $\frac{3}{8}$ (D) $\frac{1}{2}$

15. A card is drawn from a pack of 52 cards at random. The probability of getting neither an ace nor a king card is

(A) $\frac{2}{13}$ (B) $\frac{11}{13}$ (C) $\frac{4}{13}$ (D) $\frac{8}{13}$

16. The probability that a leap year will have 53 Fridays or 53 Saturdays is

(A) $\frac{2}{7}$ (B) $\frac{1}{7}$ (C) $\frac{4}{7}$ (D) $\frac{3}{7}$

17. The probability that a non-leap year will have 53 Sundays and 53 Mondays is

(A) $\frac{1}{7}$ (B) $\frac{2}{7}$ (C) $\frac{3}{7}$ (D) 0

18. The probability of selecting a queen of hearts when a card is drawn from a pack of 52 playing cards is

(A) $\frac{1}{52}$ (B) $\frac{16}{52}$ (C) $\frac{1}{13}$ (D) $\frac{1}{26}$

19. Probability of sure event is

(A) 1 (B) 0 (C) 100 (D) 0.1

20. The outcome of a random experiment results in either success or failure. If the probability of success is twice the probability of failure, then the probability of success is

(A) $\frac{1}{3}$ (B) $\frac{2}{3}$ (C) 1 (D) 0

Answers

1. SETS AND FUNCTIONS

Exercise 1.1

2. (i) A (ii) ϕ **3.** (i) {b, c} (ii) ϕ (iii) $\{a, e, f, s\}$

4. (i) {2, 4, 6, 7, 8, 9} (ii) {4, 6} (iii) {4, 6, 7, 8, 9}

10. {–5, –3, –2}, {–5, –3}, not associative

Exercise 1.2

2. Different answers are possible for (i) to (iv). One such answer is :

(i) $A' \cup (A \cap B)$ or $(A \backslash B)'$ (ii) $(A \cap B) \cup (A \cap C)$ (iii) $A \backslash (B \cup C)$ (iv) $(A \cap B) \backslash C$

5. (i) {12} (ii) {4, 8, 12, 20, 24, 28}

Exercise 1.3

1. 300 **2.** 430 **3.** 35 **5.** 100 **6.** 10%

7. (i) 10 (ii) 25 (iii) 15 **8.** (i) 450 (ii) 3550 (iii) 1850 **9.** 15

Exercise 1.4

1. (i) not a function (ii) function **2.** domain ={1, 2, 3, 4, 5}; range ={1, 3, 5, 7, 9}

3. (i) neither one to one nor onto (ii) constant function (iii) one-one and onto function

4. (i) not a function (ii) one-one function (iii) not a function (iv) bijective

5. $a = -2,\ b = -5,\ c = 8,\ d = -1$ **6.** range is $\left\{-\frac{1}{2}, -1, 1, \frac{1}{2}\right\}$; f is not a function from A to A

7. one-one and onto function **8.** (i) 12 and 14 (ii) 13 and 15 **9.** $a = 9,\ b = 15$

10. (i) $f = \{(5,-7),(6,-9),(7,-11),(8,-13)\}$

(ii) co-domain $= \{-11, 4, 7, -10, -7, -9, -13\}$

(iii) range $= \{-7, -9, -11, -13\}$ (iv) one-one function

11. (i) function (ii) function (iii) not a function (iv) not a function (v) function

12.

x	–1	–3	–5	–4
$f(x)$	2	1	6	3

13. $\{(6,1),(9,2),(15,4),(18,5),(21,6)\}$

x	6	9	15	18	21
$f(x)$	1	2	4	5	6

14. $\{(4,3),(6,4),(8,5),(10,6)\}$

x	4	6	8	10
$f(x)$	3	4	5	6

15. (i) 16 (ii) –32 (iii) 5 (iv) $\frac{2}{3}$ **16.** (i) 23 (ii) 34 (iii) 2

Exercise 1.5

1	2	3	4	5	6	7	8	9	10
A	C	C	A	A	B	A	B	B	B
11	12	13	14	15	16	17	18	19	20
A	B	C	D	A	D	D	B	A	C

2. SEQUENCES AND SERIES OF REAL NUMBERS

Exercise 2.1

1. (i) $-\frac{1}{3}, 0, 1$ (ii) $-27,\ 81,\ -243$ (iii) $-\frac{3}{4},\ 2,\ -\frac{15}{4}$

2. (i) $\frac{9}{17}, \frac{11}{21}$ (ii) –1536, 18432 (iii) 36, 78 (iv) –21, 57

3. 378, $\frac{25}{313}$ **4.** 195, 256 **5.** 2, 5, 15, 35, 75 **6.** 1, 1, 1, 2, 3, 5

Exercise 2.2

1. A.P : 6, 11, 16, $\cdots$; the general term is $5n+1$ **2.** common difference is –5, $t_{15} = 55$

3. $t_{29} = 3$ **4.** $t_{12} = 23\sqrt{2}$ **5.** $t_{17} = 84$ **6.** (i) 27 terms (ii) 34 terms

8. $t_{27} = 109$ **9.** $n = 10$ **10.** 7 **11.** First year : 100, $t_{15} = 2200$

12. 2560 **13.** 10, 2, –6 or –6, 2, 10 **14.** 2, 6, 10 or 10, 6, 2 **16.** A.P., ₹95,000

Exercise 2.3

1. (i) G.P. with $r = 2$ (ii) G.P. with $r = 5$ (iii) G.P. with $r = \frac{2}{3}$

(iv) G.P. with $r = \frac{1}{12}$ (v) G.P. with $r = \frac{1}{2}$ (vi) not a G.P.

2. -2^7 **3.** 2, 6, 18 , $\cdots$ **4.** $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \cdots$ **5.** (i) $n = 8$ (ii) $n = 11$ **6.** $n = 5$

7. $r = 5$ **8.** $r = \frac{5}{2}$ or $\frac{2}{5}$; the terms : $\frac{2}{5}, 1, \frac{5}{2}$. (or) $\frac{5}{2}, 1, \frac{2}{5}$. **9.** 18, 6, 2 (or) 2, 6, 18

10. 4, 2, 1 (or) 1, 2, 4 **11.** $1, 3, 9, \cdots$ (or) $9, 3, 1, \cdots$ **12.** ₹$1000\left(\frac{105}{100}\right)^{12}$ **13.** ₹$50,000\left(\frac{85}{100}\right)^{15}$

Exercise 2.4

1. (i) 2850 (ii) 7875 **2.** 1020 **3.** (i) 260 (ii) –75 **4.** (i) 1890 (ii) 50 5. –820

6. $\frac{39}{11} + \frac{40}{11} + \frac{41}{11} + \cdots$ **7.** 8 terms or 23 terms **8.** 55350 **9.** 740 **10.** 7227 **11.** 36

12. 12000 **13.** 15 days **14.** A.P., ₹37,200 **16.** 156 times **20.** 1225 bricks

Exercise 2.5

1. $s_{20} = \frac{15}{4}\left[1-\left(\frac{1}{3}\right)^{20}\right]$ **2.** $s_{27} = \frac{1}{6}\left[1-\left(\frac{1}{3}\right)^{27}\right]$ **3.** (i) 765 (ii) $\frac{5}{2}(3^{12}-1)$

4. (i) $\frac{1-(0.1)^{10}}{0.9}$ (ii) $\frac{10}{81}(10^{20}-1)-\frac{20}{9}$ **5.** (i) $n=6$ (ii) $n=6$ **6.** $\frac{75}{4}\left[1-\left(\frac{4}{5}\right)^{23}\right]$

7. $3+6+12+\cdots$ **8.** (i) $\frac{70}{81}[10^n-1]-\frac{7n}{9}$ (ii) $n-\frac{2}{3}\left[1-\left(\frac{1}{10}\right)^n\right]$

9. $s_{15} = \frac{5(4^{15}-1)}{3}$ **10.** 2nd option; number of mangoes 1023. **11.** $r=2$

Exercise 2.6

1. (i) 1035 (ii) 4285 (iii) 2550 (iv) 17395 (v) 10650 (vi) 382500

2. (i) $k=12$ (ii) $k=9$ **3.** 29241 **4.** 91 **5.** 3818 cm^2 **6.** 201825 cm^3

Exercise 2.7

1	2	3	4	5	6	7	8	9	10
B	D	C	D	D	A	B	B	B	B
11	12	13	14	15	16	17	18	19	20
B	A	B	D	A	B	B	A	C	A

3. ALGEBRA

Exercise 3.1

1. $\left(4,\frac{3}{2}\right)$ **2.** $(1,5)$ **3.** $(3,2)$ **4.** $\left(\frac{1}{3},\frac{1}{2}\right)$ **5.** $(1,5)$

6. $\left(\frac{11}{23},\frac{22}{31}\right)$ **7.** $(2,4)$ **8.** $(2,1)$ **9.** $\left(5,\frac{1}{7}\right)$ **10.** $(6,-4)$

Exercise 3.2

1. (i) $(4,3)$ (ii) $(0.4, 0.3)$ (iii) $(2,3)$ (iv) $\left(\frac{1}{2},\frac{1}{3}\right)$

2. (i) 23 , 7 (ii) ₹18,000 , ₹14,000 (iii) 42 (iv) ₹800 (v) 253cm^2 (vi) 720 km

Exercise 3.3

1. (i) $4,-2$ (ii) $\frac{1}{2},\frac{1}{2}$ (iii) $\frac{3}{2},-\frac{1}{3}$ (iv) $0,-2$

(v) $\sqrt{15},\ -\sqrt{15}$ (vi) $\frac{2}{3},1$ (vii) $\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}$ (viii) $-13, 11$

2. (i) x^2-3x+1 (ii) x^2-2x+4 (iii) x^2+4 (iv) $x^2-\sqrt{2}\,x+\frac{1}{5}$

(v) $x^2-\frac{x}{3}+1$ (vi) $x^2-\frac{x}{2}-4$ (vii) $x^2-\frac{x}{3}-\frac{1}{3}$ (viii) $x^2-\sqrt{3}\,x+2$

Exercise 3.4

1. (i) $x^2+2x-1,\ 4$ (ii) $3x^2-11x+40,\ -125$ (iii) $x^2+2x-2,\ 2$

(iv) $x^2-\frac{5}{3}x+\frac{5}{9},\ -\frac{50}{9}$ (v) $2x^3-\frac{x^2}{2}-\frac{3}{8}x+\frac{51}{32},-\frac{211}{32}$

(vi) $x^3-3x^2-8x+\frac{55}{2},\ -\frac{41}{2}$

2. $a=-6,\ b=11$, Remainder is 5 **3.** $p=-2,\ q=0$, Remainder is -10

Exercise 3.5

1. (i) $(x-1)(x+2)(x-3)$ (ii) $(x-1)(2x+3)(2x-1)$ (iii) $(x-1)(x-12)(x-10)$
   (iv) $(x-1)(4x^2-x+6)$ (v) $(x-1)(x-2)(x+3)$ (vi) $(x+1)(x+2)(x+10)$
   (vii) $(x-2)(x-3)(2x+1)$ (viii) $(x-1)(x^2+x-4)$ (ix) $(x-1)(x+1)(x-10)$
   (x) $(x-1)(x+6)(2x+1)$ (xi) $(x-2)(x^2+3x+7)$ (xii) $(x+2)(x-3)(x-4)$

Exercise 3.6

1. (i) $7x^2yz^3$ (ii) x^2y (iii) $5c^3$ (iv) $7xyz^2$
2. (i) $c-d$ (ii) $x-3a$ (iii) $m+3$ (iv) $x+11$ (v) $x+2y$
   (vi) $2x+1$ (vii) $x-2$ (viii) $(x-1)(x^2+1)$ (ix) $4x^2(2x+1)$ (x) $(a-1)^3(a+3)^2$
3. (i) x^2-4x+3 (ii) $x+1$ (iii) $2(x^2+1)$ (iv) x^2+4

Exercise 3.7

1. x^3y^2z 2. $12x^3y^3z$ 3. $a^2b^2c^2$ 4. $264a^4b^4c^4$ 5. a^{m+3}
6. $xy(x+y)$ 7. $6(a-1)^2(a+1)$ 8. $10xy(x+3y)(x-3y)(x^2-3xy+9y^2)$
9. $(x+4)^2(x-3)^3(x-1)$ 10. $420x^3(3x+y)^2(x-2y)(3x+1)$

Exercise 3.8

1. (i) $(x-3)(x-2)(x+6)$ (ii) $(x^2+2x+3)(x^4+2x^2+x+2)$
   (iii) $(2x^2+x-5)(x^3+8x^2+4x-21)$ (iv) $(x^3-5x-8)(2x^3-3x^2-9x+5)$
2. (i) $(x+1)(x+2)^2$ (ii) $(3x-7)^3(4x+5)$ (iii) $(x^2-y^2)(x^4+x^2y^2+y^4)$
   (iv) $x(x+2)(5x+1)$ (v) $(x-2)(x-1)$ (vi) $2(x+1)(x+2)$

Exercise 3.9

1. (i) $\dfrac{2x+3}{x-4}$ (ii) $\dfrac{1}{x^2-1}$ (iii) $(x-1)$ (iv) $\dfrac{x^2+3x+9}{x+3}$
   (v) x^2-x+1 (vi) $\dfrac{x+2}{x^2+2x+4}$ (vii) $\dfrac{x-1}{x+1}$ (viii) $(x+3)$
   (ix) $\dfrac{(x-1)}{(x+1)}$ (x) 1 (xi) $\dfrac{(x+1)}{(2x-1)}$ (xii) $(x-2)$

Exercise 3.10

1. (i) $3x$ (ii) $\dfrac{x+9}{x-2}$ (iii) $\dfrac{1}{x+4}$ (iv) $\dfrac{1}{x-1}$ (v) $\dfrac{2x+1}{x+2}$ (vi) 1
2. (i) $\dfrac{x-1}{x}$ (ii) $\dfrac{x-6}{x-7}$ (iii) $\dfrac{x+1}{x-5}$ (iv) $\dfrac{x-5}{x-11}$ (v) 1 (vi) $\dfrac{3x+1}{4(3x+4)}$ (vii) $\dfrac{x-1}{x+1}$

Exercise 3.11

1 . (i) x^2+2x+4 (ii) $\dfrac{2}{x+1}$ (iii) $\dfrac{2(x+4)}{x+3}$ (iv) $\dfrac{2}{x-5}$

(v) $\dfrac{x+1}{x-2}$ (vi) $\dfrac{4}{x+4}$ (vii) $\dfrac{2}{x+1}$ (viii) 0

2. $\dfrac{2x^3+2x^2+5}{x^2+2}$ 3. $\dfrac{5x^2-7x+6}{2x-1}$ 4. 1

Exercise 3.12

1. (i) $14|a^3b^4c^5|$ (ii) $17|(a-b)^2(b-c)^3|$ (iii) $|x-11|$

(iv) $|x+y|$ (v) $\dfrac{11}{9}\left|\dfrac{x^2}{y}\right|$ (vi) $\dfrac{8}{5}\left|\dfrac{(a+b)^2(x-y)^4(b-c)^3}{(x+y)^2(a-b)^3(b+c)^5}\right|$

2. (i) $|4x-3|$ (ii) $|(x+5)(x-5)(x+3)|$ (iii) $|2x-3y-5z|$

(iv) $\left|x^2+\dfrac{1}{x^2}\right|$ (v) $|(2x+3)(3x-2)(2x+1)|$ (vi) $|(2x-1)(x-2)(3x+1)|$

Exercise 3.13

1. (i) $|x^2-2x+3|$ (ii) $|2x^2+2x+1|$ (iii) $|3x^2-x+1|$ (iv) $|4x^2-3x+2|$
2. (i) $a=-42, b=49$ (ii) $a=12, b=9$ (iii) $a=49, b=-70$ (iv) $a=9,\ b=-12$

Exercise 3.14

1. $\{-6,3\}$ 2. $\left\{-\dfrac{4}{3},3\right\}$ 3. $\left\{-\sqrt{5},\dfrac{3}{\sqrt{5}}\right\}$ 4. $\left\{-\dfrac{3}{2},5\right\}$ 5. $\left\{-\dfrac{4}{3},2\right\}$
6. $\left\{5,\dfrac{1}{5}\right\}$ 7. $\left\{-\dfrac{5}{2},\dfrac{3}{2}\right\}$ 8. $\left\{\dfrac{1}{b^2},\dfrac{1}{a^2}\right\}$ 9. $\left\{-\dfrac{5}{2},3\right\}$ 10. $\left\{7,\dfrac{8}{3}\right\}$

Exercise 3.15

1. (i) $\{-7,1\}$ (ii) $\left\{\dfrac{-3+\sqrt{5}}{2},\dfrac{-3-\sqrt{5}}{2}\right\}$ (iii) $\left\{-3,\dfrac{1}{2}\right\}$

(iv) $\left\{\dfrac{a-b}{2},-\left(\dfrac{a+b}{2}\right)\right\}$ (v) $\{\sqrt{3},1\}$ (vi) $\{-1,3\}$

2. (i) $\{4,3\}$ (ii) $\left\{\dfrac{2}{5},\dfrac{1}{3}\right\}$ (iii) $\left\{\dfrac{1}{2},2\right\}$ (iv) $\left\{-\dfrac{2b}{3a},\dfrac{b}{a}\right\}$

(v) $\left\{\dfrac{1}{a},a\right\}$ (vi) $\left\{\dfrac{a+b}{6},\dfrac{a-b}{6}\right\}$ (vii) $\left\{\dfrac{(9+\sqrt{769})}{8},\dfrac{(9-\sqrt{769})}{8}\right\}$ (viii) $\left\{-1,\dfrac{b^2}{a^2}\right\}$

Exercise 3.16

1. 8 or $\dfrac{1}{8}$ 2. 9 and 6 3. 20 m, 5m or 10m, 10m 4. $\dfrac{3}{2}m$
5. 45km/hr 6. 5 km/hr 7. 49 years, 7 years 8. 24 cm 9. 12 days
10. Speed of the first train = 20 km / hr and the speed of the second train = 15 km / hr

Exercise 3.17

1. (i) Real (ii) Non-real (iii) Real and equal (iv) Real and equal (v) Non-real (vi) Real

2. (i) $\frac{25}{2}$ (ii) ± 3 (iii) -5 or 1 (iv) 0 or 3

Exercise 3.18

1. (i) 6,5 (ii) $-\frac{r}{k}, p$ (iii) $\frac{5}{3}, 0$ (iv) $0, -\frac{25}{8}$

2. (i) $x^2 - 7x + 12 = 0$ (ii) $x^2 - 6x + 2 = 0$ (iii) $4x^2 - 16x + 9 = 0$

3. (i) $\frac{13}{6}$ (ii) $\pm\frac{1}{3}$ (iii) $\frac{35}{18}$ **4.** $\frac{4}{3}$

5. $4x^2 - 29x + 25 = 0$ **6.** $x^2 + 3x + 2 = 0$ **7.** $x^2 - 11x + 1 = 0$

8. (i) $x^2 - 6x + 3 = 0$ (ii) $27x^2 - 18x + 1 = 0$ (iii) $3x^2 - 18x + 25 = 0$

9. $x^2 + 3x - 4 = 0$ **10.** $k = -18$ **11.** $a = \pm 24$ **12.** $p = \pm 3\sqrt{5}$

Exercise 3.19

1	2	3	4	5	6	7	8	9	10
B	C	A	A	C	D	B	C	C	C
11	12	13	14	15	16	17	18	19	20
D	B	A	A	A	D	D	D	B	C
21	22	23	24	25					
D	A	C	C	A					

4. MATRICES

Exercise 4.1

1. $\begin{pmatrix} 400 & 500 \\ 200 & 250 \\ 300 & 400 \end{pmatrix}$, $\begin{pmatrix} 400 & 200 & 300 \\ 500 & 250 & 400 \end{pmatrix}$, 3×2, 2×3 **2.** $\begin{pmatrix} 6 \\ 8 \\ 13 \end{pmatrix}$, $\begin{pmatrix} 6 & 8 & 13 \end{pmatrix}$

3. (i) 2×3 (ii) 3×1 (iii) 3×3 (iv) 1×3 (v) 4×2

4. 1×8, 8×1, 2×4, 4×2

5. $1\times 30, 30\times 1, 2\times 15, 15\times 2, 3\times 10, 10\times 3, 5\times 6, 6\times 5$.

6. (i) $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ (ii) $\begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix}$ (iii) $\begin{pmatrix} 0 & -\frac{1}{3} \\ \frac{1}{3} & 0 \end{pmatrix}$ **7.** (i) $\begin{pmatrix} 1 & \frac{1}{2} \\ 2 & 1 \\ 3 & \frac{3}{2} \end{pmatrix}$ (ii) $\begin{pmatrix} \frac{1}{2} & \frac{9}{2} \\ 0 & 2 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ (iii) $\begin{pmatrix} \frac{1}{2} & 2 \\ \frac{1}{2} & 1 \\ \frac{3}{2} & 0 \end{pmatrix}$

8. (i) 3×4 (ii) 4, 0 (iii) 2^{nd} row and 3^{rd} column **9.** $A^T = \begin{pmatrix} 2 & 4 & 5 \\ 3 & 1 & 0 \end{pmatrix}$

Exercise 4.2

1. $x = 2,\ y = -4,\ z = -1$ **2.** $x = 4,\ y = -3$

3. $\begin{pmatrix} -1 & 2 \\ 16 & -6 \end{pmatrix}$ **4.** $\begin{pmatrix} 14 & 3 \\ 14 & 5 \end{pmatrix}$ **5.** $\begin{pmatrix} 0 & -18 \\ 33 & -45 \end{pmatrix}$ **6.** $a = 3,\ b = -4$

7. $X = \begin{pmatrix} \frac{2}{5} & -\frac{12}{5} \\ -\frac{11}{5} & 3 \end{pmatrix},\ Y = \begin{pmatrix} \frac{2}{5} & \frac{13}{5} \\ \frac{14}{5} & -2 \end{pmatrix}$ **8.** $x = -3, -3,\ \ y = -1, 4$

11. $\begin{array}{c} \\ \\ \\ \end{array}\begin{array}{cccc} \text{TV} & \text{DVD} & \text{Video} & \text{CD} \end{array}$
$$\begin{pmatrix} 55 & 27 & 20 & 16 \\ 72 & 30 & 25 & 27 \\ 47 & 33 & 18 & 22 \end{pmatrix}\begin{array}{l} \text{store I} \\ \text{store II} \\ \text{store III} \end{array}$$

12. $\begin{array}{cc} \text{child} & \text{adult} \end{array}$
$$\begin{pmatrix} 5 & 5 \\ 10 & 10 \end{pmatrix}\begin{array}{l} \text{Before 2.00p.m.} \\ \text{After 2.00p.m.} \end{array}$$

Exercise 4.3

1. (i) 4×2 (ii) not defined (iii) 3×5 (iv) 2×2

2. (i) (6) (ii) $\begin{pmatrix} 8 & -11 \\ 22 & 12 \end{pmatrix}$ (iii) $\begin{pmatrix} -40 & 64 \\ 22 & 1 \end{pmatrix}$ (iv) $\begin{pmatrix} 12 & -42 \\ -6 & 21 \end{pmatrix}$

3. $\begin{pmatrix} 1750 \\ 1600 \\ 1650 \end{pmatrix}\begin{array}{l} \text{I day} \\ \text{II day} \\ \text{III day} \end{array},\ (5000)$ **4.** $x = 3,\ y = 0$ **5.** $x = 2,\ y = -5$

7. $AB = \begin{pmatrix} 15 & 4 \\ 12 & 0 \end{pmatrix},\ BA = \begin{pmatrix} 9 & 6 \\ 17 & 6 \end{pmatrix},\ AB \neq BA$ **11.** $x = -3, 5$

Exercise 4.4

1	2	3	4	5	6	7	8	9	10
D	D	A	D	B	D	B	C	C	A
11	12	13	14	15	16	17	18	19	20
B	D	D	B	C	B	A	C	B	D

5. COORDINATE GEOMETRY

Exercise 5.1

1. (i) (–2, 1) (ii) (0,2) **2.** (i) (5,–2) (ii) (2, –1) **3.** (–12, 8)

4. (2, –2) **6.** (–24,–2) **7.** (–2, 3) **8.** (–6,–3) **9.** (–1, 0), (–4, 2)

10. $\left(-3, \frac{3}{2}\right),\ (-2, 3),\ \left(-1, \frac{9}{2}\right)$ **11.** 4 : 7 internally

12. 5:2 internally, $\left(0, \frac{17}{7}\right)$ **13.** $\frac{\sqrt{130}}{2},\ \sqrt{13},\ \frac{\sqrt{130}}{2}$

Exercise 5.2

1. (i) 3 sq. units (ii) 32 sq. units (iii) 19 sq. units

2. (i) $a = -3$ (ii) $a = \frac{13}{2}$ (iii) $a = 1, 3$

3. (i) collinear (ii) not collinear (iii) collinear

4. (i) k = 1 (ii) k = 2 (iii) $k = \frac{7}{3}$

5. (i) 17 sq. units (ii) 43 sq. units (iii) 60.5 sq. units **7.** 1 sq. units, 1 : 4

Exercise 5.3

1. (i) 45° (ii) 60° (iii) 0° **2.** (i) $\frac{1}{\sqrt{3}}$ (ii) $\sqrt{3}$ (iii) undefined

3. (i) 1 (ii) –2 (iii) 1 **4.** (i) 45° (ii) 30° (iii) $\tan\theta = \frac{b}{a}$

5. $-\frac{1}{2}$ **6.** (i) 0 (ii) undefined (iii) 1 **7.** $\sqrt{3}$, 0 **10.** $a = -1$

11. $b = 6$ **12.** $-\frac{9}{10}$ **13.** $\frac{11}{7}, -13, -\frac{1}{4}$ **14.** $\frac{1}{12}, -\frac{4}{5}, \frac{9}{2}$

Exercise 5.4

1. $y = 5$, $y = -5$ **2.** $y = -2$, $x = -5$ **3.** (i) $3x + y - 4 = 0$ (ii) $\sqrt{3}x - y + 3 = 0$

4. $x - 2y + 6 = 0$ **5.** (i) slope 1, y-intercept 1 (ii) slope $\frac{5}{3}$, y-intercept 0

(iii) slope 2, y-intercept $\frac{1}{2}$ (iv) slope $-\frac{2}{3}$, y-intercept $-\frac{2}{5}$

6. (i) $4x + y - 6 = 0$ (ii) $2x - 3y - 22 = 0$ **7.** $2x - 2\sqrt{3}y + (3\sqrt{3} - 7) = 0$

8. (i) $x - 5y + 27 = 0$ (ii) $x + y + 6 = 0$ **9.** $6x + 5y - 2 = 0$

11. (i) $3x + 2y - 6 = 0$ (ii) $9x - 2y + 3 = 0$ (iii) $15x - 8y - 6 = 0$

12. (i) 3,5 (ii) –8, 16 (iii) $-\frac{4}{3}, -\frac{2}{5}$, **13.** $2x + 3y - 18 = 0$

14. $2x + y - 6 = 0$, $x + 2y - 6 = 0$ **15.** $x - y - 8 = 0$

16. $x + 3y - 6 = 0$ **17.** $2x + 3y - 12 = 0$ **18.** $x + 2y - 10 = 0$, $6x + 11y - 66 = 0$

19. $x + y - 5 = 0$ **20.** $3x - 2y + 4 = 0$

Exercise 5.5

1. (i) $-\frac{3}{4}$ (ii) 7 (iii) $\frac{4}{5}$ **4.** $a = 6$ **5.** $a = 5$ **6.** $p = 1,2$ **7.** $h = \frac{22}{9}$

8. $3x - y - 5 = 0$ **9.** $2x + y = 0$ **10.** $2x + y - 5 = 0$ **11.** $x + y - 2 = 0$

12. $5x + 3y + 8 = 0$ **13.** $x + 3y - 7 = 0$ **14.** $x - 3y + 6 = 0$

15. $x - 4y + 20 = 0$ **16.** (3, 2) **17.** 5 units **18.** $x + 2y - 5 = 0$

19. $2x + 3y - 9 = 0$

Exercise 5.6

1	2	3	4	5	6	7	8	9	10	11	12
C	B	A	D	A	B	D	A	D	C	C	B
13	14	15	16	17	18	19	20	21	22	23	
C	C	C	D	B	B	D	A	A	B	B	

6. GEOMETRY

Exercise 6.1

1. (i) 20cm (ii) 6cm (iii) 1 **2.** 7.5cm **3.** (i) No (ii) Yes **4.** 10.5cm

6. 12cm, 10cm **9.** (i) 7.5cm (ii) 5.8cm (iii) 4 cm **10.** (i) Yes (ii) No **11.** 18 cm

Exercise 6.2

1. (i) $x = 4\text{cm}$, $y = 9\text{cm}$ (ii) $x = 3.6\text{cm}$, $y = 2.4\text{cm}$, $z = 10\text{cm}$ (iii) $x = 8.4\text{cm}$, $y = 2.5\text{cm}$

2. 3.6m **3.** 1.2m **4.** 140m **6.** 6cm **7.** 64cm^2 **8.** 166.25 cm

9. (i) $\frac{9}{64}$ (ii) $\frac{55}{64}$ **10.** 6.3km^2 **11.** 72cm **12.** 9m

13. (i) $\triangle XWY$, $\triangle YWZ$, $\triangle XYZ$ (ii) 4.8m

Exercise 6.3

1. 65° **2.** (i) 4 cm (ii) 12 cm **3.** (i) 12 cm (ii) 5 cm **6.** 30 cm

Exercise 6.4

1	2	3	4	5	6	7	8	9	10
B	B	A	D	B	C	B	D	B	B
11	12	13	14	15	16	17	18	19	20
D	D	C	D	D	A	B	B	D	C

7. TRIGONOMETRY

Exercise 7.1

1. (i) No (ii) No

Exercise 7.2

1. 1.8m **2.** 30° **3.** No **4.** 174.7 m **5.** 40 cm **6.** Crow B

7. $5\sqrt{6}$ m **8.** 1912.40m **9.** $30\sqrt{2}$ m **10.** 1.098 m **11.** $19\sqrt{3}$ m

12. Yes **13.** 87m **14.** 3 Minutes **15.** 3464 km **16.** 40 m

17. 60 m ; $40\sqrt{3}$ m **18.** 90m

Exercise 7.3

1	2	3	4	5	6	7	8	9	10
B	C	C	A	A	B	A	A	C	B
11	12	13	14	15	16	17	18	19	20
B	C	A	D	C	C	D	B	B	D

8. MENSURATION

Exercise 8.1

1. 704cm^2, 1936 cm^2 2. $h = 8\text{ cm}$, 352cm^2 3. $h = 40\text{ cm}$, $d = 35\text{ cm}$
4. ₹2640 5. $r = 3.5\text{ cm}$, $h = 7\text{ cm}$ 6. $h = 28\text{ cm}$
7. $C_1 : C_2 = 5:2$ 8. $1300\pi\text{ cm}^2$ 9. 3168 cm^2
10. 550cm^2, 704 cm^2 11. $h = 15\sqrt{3}\text{ cm}$, $l = 30\text{ cm}$ 12. 1416cm^2
13. 23.1m^2 14. 10.5 cm 15. $301\frac{5}{7}\text{ cm}^2$ 16. 2.8 cm
17. 4158cm^2 18. $C_1 : C_2 = 9:25$, $T_1 : T_2 = 9:25$
19. $44.1\pi\text{ cm}^2$, $57.33\pi\text{ cm}^2$ 20. ₹246.40

Exercise 8.2

1. 18480 cm^3 2. 38.5 litres 3. 4620 cm^3 4. $r = 2.1\text{ cm}$
5. $V_1 : V_2 = 20:27$ 6. 10 cm 7. 4158 cm^3 8. 7.04 cm^3
9. 8800cm^3 10. 616cm^3 11. 5cm 12. 1408.6 cm^3
13. $314\frac{2}{7}\text{ cm}^3$ 14. $2\sqrt{13}\text{ cm}$ 15. 8 cm 16. 2.29 Kg
17. $3050\frac{2}{3}\text{ cm}^3$ 18. $288\pi\text{cm}^2$ 19. $718\frac{2}{3}\text{cm}^3$ 20. 1: 8

Exercise 8.3

1. $11.88\pi\text{ cm}^2$ 2. 7623cm^3 3. 220mm^2 4. 1034 sq.m
5. 12 cm 6. 12.8 km 7. 2 cm 8. 1 cm
9. 1386 litres 10. 3 hrs. 12 mins. 11. 16 cm 12. 16 cm
13. 750 lead shots 14. 10 cones 15. 70 cm
16. $r = 36\text{ cm}$, $l = 12\sqrt{13}\text{ cm}$ 17. 11m

Exercise 8.4

1	2	3	4	5	6	7	8	9	10	11
B	C	A	A	B	C	A	B	D	C	C
12	13	14	15	16	17	18	19	20	21	22
D	D	B	D	B	C	B	D	A	D	C

10. GRAPH

Exercise 10.1

2. (i) $\{-2, 2\}$ (ii) $\{-2, 5\}$ (iii) $\{5, 1\}$ (iv) $\left\{-\frac{1}{2}, 3\right\}$
3. $\{-1, 5\}$ 4. $\{-2, 3\}$ 5. $\{-2.5, 2\}$ 6. $\{-3, 5\}$ 7. No real solutions

Exercise 10.2

1. 120 kms 2. (i) ₹105 (ii) 11 note books 3. (i) $y = 8$ (ii) $x = 6$
4. (i) $k = 15$ (ii) ₹45 5. $y = 4$; $x = 2$ 6. 24 days

11. STATISTICS

Exercise 11.1

1. (i) 36, 0.44 (ii) 44, 0.64 **2.** 71 **3.** 3.38 kg **4.** $2\sqrt{5}$, 20

5. 3.74 **6.** (i) 5.97 (ii) 4.69 **7.** 6.32 **8.** 1.107 **9.** 15.08

10. 36.76, 6.06 **11.** 416, 20. 39 **12.** 54.19 **13.** 4800, 240400 **14.** 10.2, 1.99

15. 25 **16.** 20.41 **17.** 12 **18.** 5.24 **19.** 1159, 70

20. *A* is more consistent

Exercise 11.2

1	2	3	4	5	6	7	8	9	10
D	A	C	B	D	C	C	B	A	D
11	12	13	14	15					
D	B	C	D	B					

12. PROBABILITY

Exercise 12.1

1. $\frac{1}{10}$ **2.** $\frac{1}{9}$ **3.** $\frac{1}{3}$ **4.** $\frac{1}{5}$ **5.** $\frac{3}{4}$

6. (i) $\frac{1}{4}$ (ii) $\frac{3}{4}$ (iii) $\frac{12}{13}$ **7.** (i) $\frac{7}{8}$ (ii) $\frac{3}{8}$ (iii) $\frac{1}{2}$

8. (i) $\frac{1}{2}$ (ii) $\frac{3}{5}$ **9.** (i) $\frac{1}{10}$ (ii) $\frac{24}{25}$ **10.** $\frac{1}{2}$ **11.** (i) $\frac{1}{4}$ (ii) $\frac{2}{3}$

12. (i) $\frac{1}{4}$ (ii) $\frac{17}{20}$ **13.** $\frac{1}{3}$ **14.** $\frac{1}{36}$ **15.** $\frac{1}{6}$ **16.** 12

17. (i) $\frac{22}{25}$ (ii) $\frac{24}{25}$ **18.** (i) $\frac{1}{4}$ (ii) 3 **19.** (i) $\frac{5}{9}$ (ii) $\frac{17}{18}$

Exercise 12. 2

1. $\frac{4}{5}$ **2.** $\frac{3}{20}$ **3.** (i) $\frac{1}{5}$ (ii) $\frac{4}{5}$ **4.** $\frac{5}{9}$ **5.** $\frac{8}{25}$

6. $\frac{5}{8}$ **7.** $\frac{4}{9}$ **8.** $\frac{9}{10}$ **9.** $\frac{3}{5}$ **10.** $\frac{4}{13}$

11. $\frac{8}{13}$ **12.** $\frac{2}{3}$ **13.** $\frac{5}{13}, \frac{4}{13}$ **14.** (i) 0.45 (ii) 0.3 **15.** $\frac{101}{105}$

Exercise 12. 3

1	2	3	4	5	6	7	8	9	10
C	**D**	**B**	**A**	**A**	**B**	**A**	**A**	**D**	**A**
11	12	13	14	15	16	17	18	19	20
D	**C**	**C**	**B**	**B**	**D**	**D**	**A**	**A**	**B**

Miscellaneous problems

(Not for examination)

1. If $f(x) = \dfrac{x-1}{x+1}, x \neq -1,$ then prove that $f(2x) = \dfrac{3f(x)+1}{f(x)+3}$.

2. Solve the equation $(x-1)(x-2)(x-3)(x-4) = 15$ for real values of x.
   $\left(\text{Ans: } x = \dfrac{5 \pm \sqrt{21}}{2}\right)$

3. For what values of x do the three numbers $\log_{10} 2$, $\log_{10}(2^x - 1)$ and $\log_{10}(2^x + 3)$ taken in that order constitute an A.P.? $(\text{Ans: } x = \log_5 2)$

4. In a G.P. with common ratio r, the sum of first four terms is equal to 15 and the sum of their squares is equal to 85. Prove that $14r^4 - 17r^3 - 17r^2 - 17r + 14 = 0$.

5. Prove that the sequence $\{b_n\}$ is a G.P. if and only if $b_n^2 = b_{n-1}\, b_{n+1}, n > 1$.

6. Certain numbers appear in both arithmetic progressions 17, 21, $\cdots$ and 16, 21, $\cdots$. Find the sum of the first ten numbers appearing in both progressions.

7. Prove that the sequence $\{a_n\}$ is an A.P. if and only if $a_n = \dfrac{a_{n-1} + a_{n+1}}{2}, n > 1$.

8. Prove that $\sin^6\alpha + \cos^6\alpha + 3\sin^2\alpha\cos^2\alpha = 1$

9. Prove that $\dfrac{\sin x + \cos x}{\cos^2 x} = \tan^3 x + \tan^2 x + \tan x + 1$.

10. If we divide a two-digit number by the sum of its digits, we get 4 as a quotient and 3 as a remainder. Now if we divide that two-digit number by the product of its digits, we get 3 as a quotient and 5 as a remainder. Find the two-digit number. (Ans : 23)

11. Find the sum of all two-digit numbers which, being divided by 4, leave a remainder of 1. (Ans : 1210)

12. Simplify the expression $\dfrac{\dfrac{1}{a} + \dfrac{1}{b+c}}{\dfrac{1}{a} - \dfrac{1}{b+c}} \times \left(1 + \dfrac{b^2 + c^2 - a^2}{2bc}\right)(a+b+c)^{-2}$
    $\left(\text{Ans : } \dfrac{1}{2bc}\right)$

13. The quadratic equation $ax^2 + bx + c = 0$ has no real roots and $a + b + c < 0$. Find the sign of the number c. (Hint. If $f(x) = 0$ has no real roots, then $f(x)$ has same sign for all x) (Ans: $c < 0$)

14. Find all real numbers x such that $f(x) = \dfrac{x-1}{x^2 - x + 6} > 0$. (Ans $x > 1$)

15. Solve the equation $1 + a + a^2 + \cdots + a^x = (1+a)(1+a^2)(1+a^4)(1+a^8)$
    (Ans: $x = 15$)

16. Compute $\dfrac{6x_1^2 x_2 - 4x_1^3 + 6x_1 x_2^2 - 4x_2^3}{3x_1^2 + 5x_1 x_2 + 3x_2^2}$, where x_1 and x_2 are the roots of the equation $x^2 - 5x + 2 = 0$. $\left(\text{Ans : } -\dfrac{320}{73}\right)$

17. Prove the identity: $\operatorname{cosec}\alpha - \cot\alpha - \dfrac{\sin\alpha + \cos\alpha}{\cos\alpha} + \dfrac{\sec\alpha - 1}{\sin\alpha} = -1$

18. One-fourths of a herd of camels was seen in the forest. Twice the square root of the number of herd had gone to mountains and the remaining 15 camels were seen on the bank of a river. Find the total number of camels. (Ans: Number of camels is 36)

19. After covering a distance of 30 km with a uniform speed there is some defect in a train engine and therefore, its speed is reduced to $\frac{4}{5}$ of its original speed. Consequently, the train reaches its destination late by 45 minutes. Had it happened after covering 18 kilo metres more, the train would have reached 9 minutes earlier. Find the speed of the train and the distance of journey. (Ans: Speed of the train is 30 km/hr and the distance of the journey is 120 km.)

20. If $\sin\theta + \sin^2\theta + \sin^3\theta = 1$, then prove that $\cos^6\theta - 4\cos^4\theta + 8\cos^2\theta = 4$

21. If $\operatorname{cosec}\theta - \sin\theta = l$ and $\sec\theta - \cos\theta = m$, prove that $l^2m^2(l^2 + m^2 + 3) = 1$

22. At the foot of a mountain the elevation of its summit is 45°; after ascending 1000 m towards the mountain up a slope of 30° inclination, the elevation is found to be 60°. Find the height of the mountain. (Ans: 1.366 km)

23. If the opposite angular points of a square are (3, 4) and (1, –1), then find the coordinates of the remaining angular points. (Ans: $\left(\frac{9}{2}, \frac{1}{2}\right)$ and $\left(-\frac{1}{2}, \frac{5}{2}\right)$)

24. In an increasing G.P. the sum of first and the last term is 66, the product of the second and the last but one is 128 and the sum of the terms is 126. How many terms are there in the progression. (Ans : 6)

25. A tower subtends an angle α at a point A in the plane of its base and the angle of depression of the foot of the tower at a height b just above A is β. Prove that the height of the tower is $b\cot\beta\tan\alpha$.

26. A rectangular pool has the dimensions 40 ft × 20 ft. We have exactly 99 cu.ft of concrete to be used to create a border of uniform width and depth around the pool. If the border is to have a depth of 3 inches and if we use all of the concrete, how wide the border will be ? (Ans : 3 ft)

27. Simplify $(1 + \frac{2}{2})(1 + \frac{2}{3})(1 + \frac{2}{4})\cdots(1 + \frac{2}{n})$. (Ans : $\frac{(n + 1)(n + 2)}{6}$)

28. There are three circular disks such that two of them has radius r inches and the third has radius $2r$ inches. These three disks are placed in a plane such that each of its boundary has exactly one point in common with any other boundary. Find the area of the triangle formed by the centers of these disks. (Ans : $2\sqrt{2}\,r^2$ sq.inches)

29. Six circular discs each having radius 8 inches are placed on the floor in a circular fashion so that in the center area we could place a seventh disk touching all six of these disks exactly at one point each and each disk is touching two other disks one point each on both sides. Find the area formed by these six disks in the center. (Ans : $192\sqrt{3}$ sq. inches)

30. From a cylinderical piece of wood of radius 4 cm and height 5cm, a right circular cone with same base radius and height 3 cms is carved out.Prove that the total surface area of the remaining wood is 76π cm^2.

31. Show that $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n + 1)!} = 1 - \frac{1}{(n + 1)!}$ where $n! = 1 \times 2 \times 3 \times \cdots \times n$.

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QUESTION PAPER DESIGN

Subject : Mathematics | **Time: 2.30 Hrs**

Class : X | **Max marks: 100**

Weightage of marks to Learning Objectives

Objectives	Percentage
Knowledge	19
Understanding	31
Application	23
Skill	27
Total	100

Weightage to the types of Question

Type of Questions	Section-A Very Short Answer (Objective)	Section-B Short Answer	Section-C Long Answer	Section-D Very Long Answer	Total
Number of Questions	15	10	9	2	36
Marks	15	20	45	20	100
Time (in minutes)	20	35	65	30	2.30 Hrs

Difficulty Level

Level	Percentage of Marks
Difficult	12
Average	28
Easy	60

Sections and Options

Sections	Question numbers		Number of Questions	Questions to be answered
	From	To		
A	1	15	15	15
B	16	30	16 30th Question is compulsory and is in ‘either’ ‘or’ type	10
C	31	45	16 45th Question is compulsory and is in ‘either’ ‘or’ type	9
D	46		2 This Question is in ‘either’ ‘or’ type	1
	47		2 This Question is in ‘either’ ‘or’ type	1

Weightage to Content

Chapter No.	Chapter	Number of Questions				Total Marks
		1 mark	2 marks	5 marks	10 marks	
1	Sets and Functions	1	2	2		15
2	Sequences and series of Real Numbers	2	1	2		14
3	Algebra	2	2	3		21
4	Matrices	1	2	1		10
5	Coordinate Geometry	2	2	2		16
6	Geometry	2	1	1		9
7	Trigonometry	2	2	1		11
8	Mensuration	1	2	2		15
9	Practical Geometry				2	20
10	Graphs				2	20
11	Statistics	1	1	1		8
12	Probability	1	1	1		8
Total		15	16	16	4	167

Distribution of Marks and Questions towards Examples, Exercises and Framed questions

	Sec A (1 mark)	Sec B (2 marks)	Sec C (5 marks)	Sec D (10 marks)	Total Marks	Percentage
From the Examples given in the Text Book	---	6 (2)	6 (5)	1 (10)	52	31
From the Exercises given in the Text Book	10 (1)	8 (2)	8 (5)	3 (10)	96	58
Framed questions from specified chapters	5 (1)	2 (2)	2 (5)	---	19	11
Total	15 (1)	16 (2)	16 (5)	4 (10)	167	100

- Numbers in brackets indicate the marks for each question.

Section - A

1. All the 15 questions numbered 1 to 15 are multiple choice questions each with 4 distractors and all are compulsory. Each question carries one mark.
2. Out of 15 questions, 10 questions are from the multiple choice questions given in the Text Book. The remaining 5 questions should be framed from the five different chapters 2, 3, 5, 6 and 7 on the basis of the Text Book theorems, results, examples and exercises.

Section - B

1. 10 questions are to be answered from the questions numbered 16 to 30. Each question carries two marks.
2. Answer any 9 questions from the first 14 questions. Question No. 30 is compulsory and is in either or type.
3. The order of the first 14 questions should be in the order of the chapters in the Text Book.
4. Out of first 14 questions, 6 questions are from the examples and 8 questions are from the exercises.
5. The two questions under question no. 30 should be framed based on the examples and problems given in the exercises from any two different chapters of 2, 3, 5 and 8.

Section - C

1. 9 questions are to be answered from the questions numbered 31 to 45. Each question carries five marks.
2. Answer any 8 questions from the first 14 questions. Question no. 45 is compulsory and is in either or type.
3. The order of the first 14 questions should be in the order of the chapters in the Text Book.
4. Out of first 14 questions, 6 questions are from the examples and 8 questions are from the exercises.
5. The two questions under question no. 45 should be framed based on the examples and problems given in the exercises from any two different chapters of 2, 3, 5 and 8.
6. Questions numbered 30(a), 30(b), 45(a) and 45(b) should be framed based on the examples and problems given in the exercises from the chapters 2, 3, 5 and 8 subject to the condition that all of them should be from different chapters.

Section - D

1. This section contains two questions numbered 46 and 47, one from the chapter 9 and the other from the chapter 10, each with two alternatives ('either' 'or' type) from the same chapter. Each question carries ten marks.
2. Answer both the questions.
3. One of the questions 46(a), 47(a), 46(b) and 47(b) should be from the examples given in the text book. The remaining three questions should be from the exercises.

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Chapter / Objective	Knowledge				Understanding				Application				Skill				Total marks
	VSA	SA	LA	VLA	VSA	SA	LA	VLA	VSA	SA	LA	VLA	VSA	SA	LA	VLA	
Sets and Functions	1(1)	2(1)	5(1)			2(1)					5(1)						15
Sequences and Series of Real Numbers		2(1)	5(1)		1(1)				1(1)		5(1)						14
Algebra		2(1)	5(1)		1(1)				1(1)	2(1)	5(1)				5(1)		21
Matrices						4(2)	5(1)		1(1)								10
Coordinate Geometry		2(1)			1(1)	2(1)	5(1)		1(1)		5(1)						16
Geometry					1(1)	2(1)	5(1)		1(1)								9
Trigonometry					1(1)	2(1)	5(1)		1(1)	2(1)							11
Mensuration	1(1)					2(1)	5(1)			2(1)	5(1)						15
Practical Geometry																10(2)	20
Graphs																10(2)	20
Statistics			5(1)			2(1)			1(1)								8
Probability		2(1)					5(1)		1(1)								8
Total	2(2)	10(5)	20(4)		5(5)	16(8)	30(6)		8(8)	6(3)	25(5)				5(1)	40(4)	167

- Numbers in brackets indicate the number of questions.
- Other numbers indicate the marks.