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+2

MATHEMATICS

STUDY MATERIAL

Chapter 1 -Applications of Matrices and Determinants

Carl Friedrich Gauss

Chapter – 1

Applications of Matrices and Determinants

EXERCISE 1.1

Concept Corner

- The determinant of submatrix is called **minor** of the element a_{ij}. It is denoted by M_{ij}.
- The product of M_{ij} and $(-1)^{i+j}$ is called **cofactor** of the element $a_{ij.}$ It is denoted by $A_{ij.}$ Thus **the cofactor of a_{ij} is $A_{ij} = (-1)^{i+j} M_{ij}$.**
- Let A be a square matrix of order n. Then the matrix of cofactors of A is defined as the matrix obtained by replacing each element a_{ij} of A with the corresponding cofactor A_{ij}.
- The **adjoint matrix of A** is defined as the transpose of the matrix of cofactors of A. It is denoted by **adj A**.
- Let A be a square matrix of order n. If there exists a square matrix B of order n such that $AB = BA = I_n$, then the matrix B is called an **inverse of A**.
- A square matrix A is called **orthogonal** if $AA^T = A^TA = I$. (A is **orthogonal** if and only if A is non-singular and $A^{-1} = A^T$)

Theorems:

1. For every square matrix A of order n, $A(adj\ A) = (adjA)A = |A|I_n$.
2. If a square matrix has an inverse, then it is unique.
3. Let A be square matrix of order n. Then, A^{-1} exists if any only if A is non – singular.
4. If A is non – singular, then

   (i) $|A^{-1}| = \frac{1}{|A|}$ (ii) $(A^T)^{-1} = (A^{-1})^T$ (iii) $(\lambda A)^{-1} = \frac{1}{\lambda} A^{-1}$, where λ is a non – zero scalar.
5. Left Cancellation Law

   Let A, B, and C be square matrices of order n. If A is non-singular and $AB = AC$, then $B = C$
6. Right Cancellation Law

   Let A, B and C be square matrices of order n. If A is non- singular and $BA = CA$, then $B = C$.
7. Reversal Law for Inverses

   If A and B are non- singular matrices of the same order, then the product AB is also non-singular and $(AB)^{-1} = B^{-1}A^{-1}$.
8. Law of Double Inverse, If A is non-singular, then A^{-1} is also non-singular and $(A^{-1})^{-1} = A$.
9. If A is a non-singular square matrix of order n, *then*

   (i) $(adj\ A)^{-1} = adj(A^{-1}) = \frac{1}{|A|} A$ (ii) $|adjA| = |A|^{n-1}$

   (iii) $adj(adjA) = |A|^{n-2} A$ (iv) $adj(\lambda A) = \lambda^{n-1} adj(A), \lambda$ is a nonzero scalar

   (v) $|adj(adjA)| = |A|^{(n-1)^2}$ (vi) $(adjA)^T = adj(A^T)$
10. If A and B are any two non-singular square matrices of order n, then

    adj $(AB) = (adjB)(adjA)$.

1. Find the adjoint of the following: (i) $\begin{bmatrix} -3 & 4 \\ 6 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$ (iii) $\frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$

2M

(i) Let $A = \begin{bmatrix} -3 & 4 \\ 6 & 2 \end{bmatrix}$

$A_{11} = \text{co-factor of } -3 = 2$

$A_{12} = \text{co-factor of } 4 = -6$

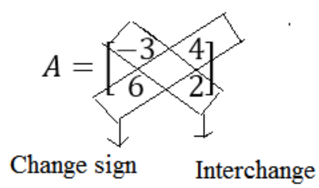
$A_{21} = \text{co-factor of } 6 = -4$

$A_{22} = \text{co-factor of } 2 = -3$

$$adjA = \begin{bmatrix} 2 & -6 \\ -4 & -3 \end{bmatrix}^T$$

$$= \begin{bmatrix} 2 & -4 \\ -6 & -3 \end{bmatrix}$$

Note:



$$adjA = \begin{bmatrix} 2 & -4 \\ -6 & -3 \end{bmatrix}$$

3M

(ii) Let $A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$

$$adjA = \begin{bmatrix} \begin{vmatrix} 4 & 1 \\ 7 & 2 \end{vmatrix} & -\begin{vmatrix} 3 & 1 \\ 3 & 2 \end{vmatrix} & \begin{vmatrix} 3 & 4 \\ 3 & 7 \end{vmatrix} \\ -\begin{vmatrix} 3 & 1 \\ 7 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 3 & 7 \end{vmatrix} \\ \begin{vmatrix} 3 & 1 \\ 4 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} (8-7) & -(6-3) & (21-12) \\ -(6-7) & (4-3) & -(14-9) \\ (3-4) & -(2-3) & (8-9) \end{bmatrix}^T$$

$$= \begin{bmatrix} 1 & -3 & 9 \\ 1 & 1 & -5 \\ -1 & 1 & -1 \end{bmatrix}^T$$

$$\therefore adj\ A = \begin{bmatrix} 1 & 1 & -1 \\ -3 & 1 & 1 \\ 9 & -5 & -1 \end{bmatrix}$$

3M

(iii) Let $B = \frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$

Let us consider $A = \begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$

$$adjA = \begin{bmatrix} \begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix} & -\begin{vmatrix} -2 & 2 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} \\ -\begin{vmatrix} 2 & 1 \\ -2 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 2 \\ 1 & -2 \end{vmatrix} \\ \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ -2 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 2 \\ -2 & 1 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} (2+4) & -(-4-2) & (4-1) \\ -(4+2) & (4-1) & -(-4-2) \\ (4-1) & -(4+2) & (2+4) \end{bmatrix}^T$$

$$= \begin{bmatrix} 6 & 6 & 3 \\ -6 & 3 & 6 \\ 3 & -6 & 6 \end{bmatrix}^T$$

$$adj\ A = \begin{bmatrix} 6 & -6 & 3 \\ 6 & 3 & -6 \\ 3 & 6 & 6 \end{bmatrix}$$

$$adj\ B = adj\left(\frac{1}{3}A\right)$$

$$= \left(\frac{1}{3}\right)^{3-1} adjA$$

$$= \frac{1}{3^2}\begin{bmatrix} 6 & -6 & 3 \\ 6 & 3 & -6 \\ 3 & 6 & 6 \end{bmatrix}$$

$$= \frac{1}{9} \times 3 \begin{bmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{bmatrix}$$

$$\therefore adj\ B = \frac{1}{3}\begin{bmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{bmatrix}$$

$\because$[If A is a matrix of order n,
then $adj(\lambda A) = \lambda^{n-1} adjA$]

2. Find the inverse (if it exists) of the following (i) $\begin{bmatrix}-2 & 4\\ 1 & -3\end{bmatrix}$ (ii) $\begin{bmatrix}5 & 1 & 1\\ 1 & 5 & 1\\ 1 & 1 & 5\end{bmatrix}$ (iii) $\begin{bmatrix}2 & 3 & 1\\ 3 & 4 & 1\\ 3 & 7 & 2\end{bmatrix}$

(i) Let $A = \begin{bmatrix}-2 & 4\\ 1 & -3\end{bmatrix}$ 2M

$|A| = 6 - 4 = 2 \neq 0.$ $\therefore A^{-1}$ exists

$adj\ A = \begin{bmatrix}-3 & -4\\ -1 & -2\end{bmatrix}$

$A^{-1} = \frac{1}{|A|} adj\ A$

$\therefore A^{-1} = \frac{1}{2}\begin{bmatrix}-3 & -4\\ -1 & -2\end{bmatrix}$

(ii) Let $A = \begin{bmatrix}5 & 1 & 1\\ 1 & 5 & 1\\ 1 & 1 & 5\end{bmatrix}$ 3M

$|A| = 5[25-1] - 1[5-1] + 1[1-5]$

$= 5(24) - 1(4) + 1(-4)$

$= 120 - 4 - 4 = 112 \neq 0$

$\therefore A^{-1}$ exists.

$$adjA = \begin{bmatrix} \begin{vmatrix}5 & 1\\ 1 & 5\end{vmatrix} & -\begin{vmatrix}1 & 1\\ 1 & 5\end{vmatrix} & \begin{vmatrix}1 & 5\\ 1 & 1\end{vmatrix} \\ -\begin{vmatrix}1 & 1\\ 1 & 5\end{vmatrix} & \begin{vmatrix}5 & 1\\ 1 & 5\end{vmatrix} & -\begin{vmatrix}5 & 1\\ 1 & 1\end{vmatrix} \\ \begin{vmatrix}1 & 1\\ 5 & 1\end{vmatrix} & -\begin{vmatrix}5 & 1\\ 1 & 1\end{vmatrix} & \begin{vmatrix}5 & 1\\ 1 & 5\end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix}(25-1) & -(5-1) & (1-5)\\ -(5-1) & (25-1) & -(5-1)\\ (1-5) & -(5-1) & (25-1)\end{bmatrix}^T$$

$$= \begin{bmatrix}24 & -4 & -4\\ -4 & 24 & -4\\ -4 & -4 & 24\end{bmatrix}^T$$

$$adj\ A = \begin{bmatrix}24 & -4 & -4\\ -4 & 24 & -4\\ -4 & -4 & 24\end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} adj\ A = \frac{1}{112}\begin{bmatrix}24 & -4 & -4\\ -4 & 24 & -4\\ -4 & -4 & 24\end{bmatrix}$$

$$A^{-1} = \frac{1}{112} \times 4\begin{bmatrix}6 & -1 & -1\\ -1 & 6 & -1\\ -1 & -1 & 6\end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{28}\begin{bmatrix}6 & -1 & -1\\ -1 & 6 & -1\\ -1 & -1 & 6\end{bmatrix}$$

(iii) Let $A = \begin{bmatrix}2 & 3 & 1\\ 3 & 4 & 1\\ 3 & 7 & 2\end{bmatrix}$ 3M

$|A| = 2(8-7) - 3(6-3) + 1(21-12)$

$= 2(1) - 3(3) + 1(9)$

$= 2 - 9 + 9$

$= 2 \neq 0$

$\therefore A^{-1}$ exists

$$adjA = \begin{bmatrix} \begin{vmatrix}4 & 1\\ 7 & 2\end{vmatrix} & -\begin{vmatrix}3 & 1\\ 3 & 2\end{vmatrix} & \begin{vmatrix}3 & 4\\ 3 & 7\end{vmatrix} \\ -\begin{vmatrix}3 & 1\\ 7 & 2\end{vmatrix} & \begin{vmatrix}2 & 1\\ 3 & 2\end{vmatrix} & -\begin{vmatrix}2 & 3\\ 3 & 7\end{vmatrix} \\ \begin{vmatrix}3 & 1\\ 4 & 1\end{vmatrix} & -\begin{vmatrix}2 & 1\\ 3 & 1\end{vmatrix} & \begin{vmatrix}2 & 3\\ 3 & 4\end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix}(8-7) & -(6-3) & (21-12)\\ -(6-7) & (4-3) & -(14-9)\\ (3-4) & -(2-3) & (8-9)\end{bmatrix}^T$$

$$= \begin{bmatrix}1 & -3 & 9\\ 1 & 1 & -5\\ -1 & 1 & -1\end{bmatrix}^T$$

$$\therefore adj\ A = \begin{bmatrix}1 & 1 & -1\\ -3 & 1 & 1\\ 9 & -5 & -1\end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} adj\ A$$

$$A^{-1} = \frac{1}{2}\begin{bmatrix}1 & 1 & -1\\ -3 & 1 & 1\\ 9 & -5 & -1\end{bmatrix}$$

3. If $\boldsymbol{F(\alpha)} = \begin{bmatrix} \boldsymbol{cos\alpha} & \boldsymbol{0} & \boldsymbol{sin\alpha} \\ \boldsymbol{0} & \boldsymbol{1} & \boldsymbol{0} \\ \boldsymbol{-sin\alpha} & \boldsymbol{0} & \boldsymbol{cos\alpha} \end{bmatrix}$, show that $[\boldsymbol{F(\alpha)}]^{-1} = \boldsymbol{F(-\alpha)}$. 5M

$$F(\alpha) = \begin{bmatrix} cos\alpha & 0 & sin\alpha \\ 0 & 1 & 0 \\ -sin\alpha & 0 & cos\alpha \end{bmatrix}$$

$$F(-\alpha) = \begin{bmatrix} \cos(-\alpha) & 0 & \sin(-\alpha) \\ 0 & 1 & 0 \\ -\sin(-\alpha) & 0 & \cos(-\alpha) \end{bmatrix}$$

$[\because \sin(-\alpha) = -\sin\alpha, \quad \cos(-\alpha) = cos\alpha]$

$$F(-\alpha) = \begin{bmatrix} cos\alpha & 0 & -sin\alpha \\ 0 & 1 & 0 \\ sin\alpha & 0 & cos\alpha \end{bmatrix} \text{.......................(1)}$$

To find $[\boldsymbol{F(\alpha)}]^{-1}$:

$$|F(\alpha)| = cos\alpha[cos\alpha - 0] - 0 + \sin\alpha[0 + \sin\alpha]$$

$$= \cos^2\alpha + \sin^2\alpha$$

$$= 1 \neq 0$$

$\therefore [F(\alpha)]^{-1}$ exists.

$$adj[F(\alpha)] = \begin{bmatrix} \begin{vmatrix} 1 & 0 \\ 0 & cos\alpha \end{vmatrix} & -\begin{vmatrix} 0 & 0 \\ -\sin\alpha & \cos\alpha \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ -\sin\alpha & 0 \end{vmatrix} \\ -\begin{vmatrix} 0 & \sin\alpha \\ 0 & \cos\alpha \end{vmatrix} & \begin{vmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{vmatrix} & -\begin{vmatrix} \cos\alpha & 0 \\ -\sin\alpha & 0 \end{vmatrix} \\ \begin{vmatrix} 0 & +\sin\alpha \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} \cos\alpha & +\sin\alpha \\ 0 & 0 \end{vmatrix} & \begin{vmatrix} \cos\alpha & 0 \\ 0 & 1 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} (cos\alpha - 0) & -(0-0) & (0 + sin\alpha) \\ -(0-0) & (\cos^2\alpha + \sin^2\alpha) & -(0-0) \\ (0 - sin\alpha) & -(0-0) & (cos\alpha - 0) \end{bmatrix}^T$$

$$= \begin{bmatrix} cos\alpha & 0 & sin\alpha \\ 0 & 1 & 0 \\ -sin\alpha & 0 & cos\alpha \end{bmatrix}^T$$

$[\because \sin^2 x + \cos^2 x = 1]$

$$adj[F(\alpha)] = \begin{bmatrix} cos\alpha & 0 & -sin\alpha \\ 0 & 1 & 0 \\ sin\alpha & 0 & cos\alpha \end{bmatrix}$$

$$[F(\alpha)]^{-1} = \frac{1}{|F(\alpha)|} adj\,[F(\alpha)]$$

$$= \frac{1}{1}\begin{bmatrix} cos\alpha & 0 & -sin\alpha \\ 0 & 1 & 0 \\ sin\alpha & 0 & cos\alpha \end{bmatrix}$$

$$\therefore [F(\alpha)]^{-1} = \begin{bmatrix} cos\alpha & 0 & -sin\alpha \\ 0 & 1 & 0 \\ sin\alpha & 0 & cos\alpha \end{bmatrix} \text{.........................(2)}$$

From (1) and (2), we get $[F(\alpha)]^{-1} = F(-\alpha)$

4. **If $A = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}$, show that $A^2 - 3A - 7I_2 = O_2$. Hence find A^{-1}.** 5M

$A^2 = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 25-3 & 15-6 \\ -5+2 & -3+4 \end{bmatrix}$

$A^2 = \begin{bmatrix} 22 & 9 \\ -3 & 1 \end{bmatrix}$

$3A = 3\begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 15 & 9 \\ -3 & -6 \end{bmatrix}$

$7I_2 = 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$

$A^2 - 3A - 7I_2 = \begin{bmatrix} 22 & 9 \\ -3 & 1 \end{bmatrix} - \begin{bmatrix} 15 & 9 \\ -3 & -6 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$

$= \begin{bmatrix} 22-15-7 & 9-9-0 \\ -3+3-0 & 1+6-7 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\therefore A^2 - 3A - 7I_2 = 0$

To find A^{-1}:

$A^2 - 3A - 7I_2 = 0$

Post – multiplying by A^{-1}, we get

$A - 3I - 7A^{-1} = 0$

$7A^{-1} = A - 3I$

$7A^{-1} = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$7A^{-1} = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} + \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix}$

$7A^{-1} = \begin{bmatrix} 2 & 3 \\ -1 & -5 \end{bmatrix}$

$A^{-1} = \frac{1}{7}\begin{bmatrix} 2 & 3 \\ -1 & -5 \end{bmatrix}$

> $A^{-1}A^{-2} = (A^{-1}A)A = IA = A$
>
> $A^{-1}3A = 3(A^{-1}A) = 3I$
>
> $A^{-1}I_2 = A^{-1}$

5. **If $A = \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$, prove that $A^{-1} = A^T$.** 3M

$A = \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$

$A^T = \frac{1}{9}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix}$ ....(1)

$|A| = \left(\frac{1}{9}\right)^3 [-8(16+56) - 1(16-7) + 4(-32-4)]$ $\quad [\because |KA| = K^n|A|]$

$= \frac{1}{729}[-8(72) - 1(9) + 4(-36)]$

$= \frac{1}{729}[-576 - 9 - 144] = \frac{1}{729}(-729)$

$|A| = -1 \neq 0$

$\therefore A^{-1}$ exists.

$$adj\ A = \left(\frac{1}{9}\right)^{3-1}\begin{bmatrix} \begin{vmatrix} 4 & 7 \\ -8 & 4 \end{vmatrix} & -\begin{vmatrix} 4 & 7 \\ 1 & 4 \end{vmatrix} & \begin{vmatrix} 4 & 4 \\ 1 & -8 \end{vmatrix} \\ -\begin{vmatrix} 1 & 4 \\ -8 & 4 \end{vmatrix} & \begin{vmatrix} -8 & 4 \\ 1 & 4 \end{vmatrix} & -\begin{vmatrix} -8 & 1 \\ 1 & -8 \end{vmatrix} \\ \begin{vmatrix} 1 & 4 \\ 4 & 7 \end{vmatrix} & -\begin{vmatrix} -8 & 4 \\ 4 & 7 \end{vmatrix} & \begin{vmatrix} -8 & 1 \\ 4 & 4 \end{vmatrix} \end{bmatrix}^T$$

$[\because adj(\lambda A) = \lambda^{n-1}(adj\ A)]$

$$= \frac{1}{81}\begin{bmatrix} (16+56) & -(16-7) & (-32-4) \\ -(4+32) & (-32-4) & -(64-1) \\ (7-16) & -(-56-16) & (-32-4) \end{bmatrix}$$

$$= \frac{1}{81}\begin{bmatrix} 72 & -9 & -36 \\ -36 & -36 & -63 \\ -9 & 72 & -36 \end{bmatrix}^T$$

$$adj\ A = \frac{1}{81}\begin{bmatrix} 72 & -36 & -9 \\ -9 & -36 & +72 \\ -36 & -63 & -36 \end{bmatrix}$$

$$= \frac{1}{81} \times 9\begin{bmatrix} 8 & -4 & -1 \\ -1 & -4 & 8 \\ -4 & -7 & -4 \end{bmatrix}$$

$$= \frac{1}{9} \times \begin{bmatrix} 8 & -4 & -1 \\ -1 & -4 & 8 \\ -4 & -7 & -4 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} adj\ A$$

$$= \frac{1}{-1} \cdot \frac{1}{9}\begin{bmatrix} 8 & -4 & -1 \\ -1 & -4 & 8 \\ -4 & -7 & -4 \end{bmatrix}$$

$$A^{-1} = \frac{1}{9}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix} \text{.................(2)}$$

From (1) and (2), we get $A^{-1} = A^T$

6. If $A = \begin{bmatrix} 8 & -4 \\ -5 & 3 \end{bmatrix}$, verify that $A(adjA) = (adjA)A = |A|I_2$. 3M

$$A = \begin{bmatrix} 8 & -4 \\ -5 & 3 \end{bmatrix}$$

$$adj\ A = \begin{bmatrix} 3 & 4 \\ 5 & 8 \end{bmatrix}$$

$$A(adj\ A) = \begin{bmatrix} 8 & -4 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 5 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 24-20 & 32-32 \\ -15+15 & -20+24 \end{bmatrix}$$

$$A(adj\ A) = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \text{...................(1)}$$

$$(adj\ A)A = \begin{bmatrix} 3 & 4 \\ 5 & 8 \end{bmatrix}\begin{bmatrix} 8 & -4 \\ -5 & 3 \end{bmatrix}$$

$$(adj\ A)A = \begin{bmatrix} 24-20 & -12+12 \\ 40-40 & -20+24 \end{bmatrix}$$

$$(adj\ A)A = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \text{...................(2)}$$

$$|A| = 24 - 20 = 4$$

$$|A|I_2 = 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \text{.............(3)}$$

From (1), (2) and (3), we get

$$A(adj\ A) = (adjA)A = |A|I_2.$$

7. If $A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & -3 \\ 5 & 2 \end{bmatrix}$, verify that $(AB)^{-1} = B^{-1}A^{-1}$ 3M

$$AB = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}\begin{bmatrix} -1 & -3 \\ 5 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} -3+10 & -9+4 \\ -7+25 & -21+10 \end{bmatrix}$$
$$AB = \begin{bmatrix} 7 & -5 \\ +18 & -11 \end{bmatrix}$$
$$|AB| = -77 + 90 = 13 \neq 0$$
$(AB)^{-1}$ exists.
$$adj\ (AB) = \begin{bmatrix} -11 & 5 \\ -18 & 7 \end{bmatrix}$$
$$(AB)^{-1} = \frac{1}{|AB|} adj\ (AB)$$
$$= \frac{1}{13}\begin{bmatrix} -11 & 5 \\ -18 & 7 \end{bmatrix} \quad \ldots\ldots\ldots\ldots(1)$$
$$A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$$
$$|A| = 15 - 14 = 1 \neq 0$$
A^{-1} exists
$$adj\ A = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$$
$$A^{-1} = \frac{1}{|A|} adj\ A$$
$$A^{-1} = \frac{1}{1}\begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$$
$$B = \begin{bmatrix} -1 & -3 \\ 5 & 2 \end{bmatrix}$$
$$|B| = -2 + 15 = 13 \neq 0$$
$\therefore B^{-1}$ exists
$$adj\ B = \begin{bmatrix} 2 & 3 \\ -5 & -1 \end{bmatrix}$$
$$B^{-1} = \frac{1}{|B|} adj\ B$$
$$B^{-1} = \frac{1}{13}\begin{bmatrix} 2 & 3 \\ -5 & -1 \end{bmatrix}$$
$$B^{-1}A^{-1} = \frac{1}{13}\begin{bmatrix} 2 & 3 \\ -5 & -1 \end{bmatrix}\begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$$
$$= \frac{1}{13}\begin{bmatrix} 10-21 & -4+9 \\ -25+7 & 10-3 \end{bmatrix}$$
$$= \frac{1}{13}\begin{bmatrix} -11 & 5 \\ -18 & 7 \end{bmatrix} \quad \ldots\ldots\ldots\ldots(2)$$
From (1) and (2), we get
$$(AB)^{-1} = B^{-1}A^{-1}$$

8. If $adj(A) = \begin{bmatrix} 2 & -4 & 2 \\ -3 & 12 & -7 \\ -2 & 0 & 2 \end{bmatrix}$, find A. 3M

$$|adj\ A| = 2[24-0] + 4[-6-4] + 2[0+24] = 48 - 80 + 48 = 16$$
$$adj(adjA) = \begin{bmatrix} \begin{vmatrix} 12 & -7 \\ 0 & 2 \end{vmatrix} & -\begin{vmatrix} -3 & -7 \\ -2 & 2 \end{vmatrix} & \begin{vmatrix} -3 & 12 \\ -2 & 0 \end{vmatrix} \\ -\begin{vmatrix} -4 & 2 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 2 \\ -2 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & -4 \\ -2 & 0 \end{vmatrix} \\ \begin{vmatrix} -4 & 2 \\ 12 & -7 \end{vmatrix} & -\begin{vmatrix} 2 & 2 \\ -3 & -7 \end{vmatrix} & \begin{vmatrix} 2 & -4 \\ -3 & 12 \end{vmatrix} \end{bmatrix}^T$$
$$= \begin{bmatrix} (24-0) & -(-6-14) & (0+24) \\ -(-8-0) & (4+4) & -(0-8) \\ (28-24) & -(-14+6) & (24-12) \end{bmatrix}^T$$
$$= \begin{bmatrix} 24 & 20 & 24 \\ 8 & 8 & 8 \\ 4 & 8 & 12 \end{bmatrix}^T = \begin{bmatrix} 24 & 8 & 4 \\ 20 & 8 & 8 \\ 24 & 8 & 12 \end{bmatrix}$$
$$A = \pm\frac{1}{\sqrt{|adj\ A|}} adj\ (adj\ A) = \pm\frac{1}{\sqrt{16}}\begin{bmatrix} 24 & 8 & 4 \\ 20 & 8 & 8 \\ 24 & 8 & 12 \end{bmatrix} = \pm\frac{1}{4} \times 4\begin{bmatrix} 6 & 2 & 1 \\ 5 & 2 & 2 \\ 6 & 2 & 3 \end{bmatrix}$$
$$\therefore A = \pm\begin{bmatrix} 6 & 2 & 1 \\ 5 & 2 & 2 \\ 6 & 2 & 3 \end{bmatrix}$$

9. If $\boldsymbol{adj(A)} = \begin{bmatrix} \mathbf{0} & \mathbf{-2} & \mathbf{0} \\ \mathbf{6} & \mathbf{2} & \mathbf{-6} \\ \mathbf{-3} & \mathbf{0} & \mathbf{6} \end{bmatrix}$, find $\boldsymbol{A^{-1}}$. PTA-6 2M

$$adj(A) = \begin{bmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{bmatrix}$$

$$|adjA| = 0 + 2[36 - 18] + 0$$

$$|adjA| = 36$$

$$A^{-1} = \pm\frac{1}{\sqrt{|adj\ A|}}\ (adj\ A)$$

$$= \pm\frac{1}{\sqrt{36}}\begin{bmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{bmatrix}$$

$$= \pm\frac{1}{\sqrt{6\times 6}}\begin{bmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{bmatrix}$$

$$A^{-1} = \pm\frac{1}{6}\begin{bmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{bmatrix}$$

10. Find $\boldsymbol{adj(adj(A)}$ if adj $\boldsymbol{A} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{2} & \mathbf{0} \\ \mathbf{-1} & \mathbf{0} & \mathbf{1} \end{bmatrix}$ 3M

$$\text{adj } A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$adj(adjA) = \begin{bmatrix} \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & 0 \\ -1 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 2 \\ -1 & 0 \end{vmatrix} \\ -\begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ -1 & 0 \end{vmatrix} \\ \begin{vmatrix} 0 & 1 \\ 2 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} (2-0) & -(0+0) & (0+2) \\ -(0-0) & (1+1) & -(0+0) \\ (0-2) & -(0-0) & (2-0) \end{bmatrix}^T$$

$$= \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ -2 & 0 & 2 \end{bmatrix}^T$$

$$\therefore\ adj\ (adjA) = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

11. $A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$, **show that** $A^T A^{-1} = \begin{bmatrix} \cos 2x & -\sin 2x \\ \sin 2x & \cos 2x \end{bmatrix}$ 3M

$$A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$$

$$|A| = 1 + \tan^2 x = \sec^2 x$$

$$adj\ A = \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} adj\ A$$

$$A^{-1} = \frac{1}{\sec^2 x}\begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$$

$$A^T A^{-1} = \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix} \frac{1}{\sec^2 x}\begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$$

$$= \frac{1}{\sec^2 x}\begin{bmatrix} 1 - \tan^2 x & -2\tan x \\ 2\tan x & 1 - tan^2 x \end{bmatrix}$$

$$= \cos^2 x \begin{bmatrix} 1 - \frac{\sin^2 x}{\cos^2 x} & -\frac{2\sin x}{\cos x} \\ 2\frac{\sin x}{\cos x} & 1 - \frac{\sin^2 x}{\cos^2 x} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\cos^2 x - \sin^2 x}{\cos^2 x}\cos^2 x & \frac{-2\sin x \cos^2 x}{\cos x} \\ \frac{2\sin x}{\cos x}\cos^2 x & \frac{\cos^2 x - \sin^2 x}{\cos^2 x}\cos^2 x \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2 x - \sin^2 x & -2\sin x \cos x \\ 2\sin x \cos x & \cos^2 x - \sin^2 x \end{bmatrix} \qquad \begin{bmatrix} \because \cos^2 x - \sin^2 x = cos 2x \\ 2sinx \cos x = \sin 2x \end{bmatrix}$$

$$A^T A^{-1} = \begin{bmatrix} \cos 2x & -sin 2x \\ \sin 2x & \cos 2x \end{bmatrix}$$

Hence proved

12. Find the matrix A **for which** $A\begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 14 & 7 \\ 7 & 7 \end{bmatrix}$. 3M

Let $B = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}$ and $C = \begin{bmatrix} 14 & 7 \\ 7 & 7 \end{bmatrix}$

Then $AB = C$

Post multiplying by B^{-1}, we get

$$A(BB^{-1}) = CB^{-1}$$

$$AI = CB^{-1}$$

$$A = CB^{-1} \qquad \text{..................(1)}$$

$|B| = -10 + 3 = -7 \neq 0$

$\therefore B^{-1}$ exists

$$adjB = \begin{bmatrix} -2 & -3 \\ 1 & 5 \end{bmatrix}$$

$$B^{-1} = \frac{1}{|B|} adj\ B$$

$$= \frac{1}{-7}\begin{bmatrix} -2 & -3 \\ 1 & 5 \end{bmatrix}$$

$$= \frac{1}{7}\begin{bmatrix} 2 & 3 \\ -1 & -5 \end{bmatrix}$$

$$(1) \Rightarrow A = \begin{bmatrix} 14 & 7 \\ 7 & 7 \end{bmatrix} \frac{1}{7}\begin{bmatrix} 2 & 3 \\ -1 & -5 \end{bmatrix}$$

$$= \frac{1}{7} \times 7 \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -1 & -5 \end{bmatrix}$$

$$A = \begin{bmatrix} 4 - 1 & 6 - 5 \\ 2 - 1 & 3 - 5 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 \\ 1 & -2 \end{bmatrix}$$

13. Given $A = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$, find a matrix X such that $AXB = C$ 3M

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$$

$$|A| = 0 + 2 = 2 \neq 0$$

$\therefore A^{-1}$ exists

$$adjA = \begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} adj\ A$$

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$$

$$|B| = 3 + 2 = 5 \neq 0$$

$\therefore B^{-1}$ exists

$$adj\ B = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$$

$$B^{-1} = \frac{1}{|B|} adj\ B$$

$$B^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$$

Given: $AXB = C$

Premultiplying by A^{-1}, we get

$$(A^{-1}A)XB = A^{-1}C$$

$$(IX)B = A^{-1}C$$

$$XB = A^{-1}C$$

Post – multiplying by B^{-1}, we get

$$X(BB^{-1}) = A^{-1}CB^{-1}$$

$$XI = A^{-1}CB^{-1}$$

$$X = A^{-1}(CB^{-1})$$

$$X = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\frac{1}{5}\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$$

$$X = \frac{1}{10}\begin{bmatrix} 0+2 & 0+2 \\ -2+2 & -2+2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$$

$$X = \frac{1}{10}\begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$$

$$X = \frac{1}{10}\begin{bmatrix} 2-2 & 4+6 \\ 0+0 & 0+0 \end{bmatrix}$$

$$= \frac{1}{10}\begin{bmatrix} 0 & 10 \\ 0 & 0 \end{bmatrix}$$

$$\therefore X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

14. If $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, show that $A^{-1} = \frac{1}{2}(A^2 - 3I)$. 3M

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$|A| = 0 - 1[0-1] + 1[1-0] = 1 + 1 = 2 \neq 0$$

$\therefore A^{-1}$ exists

$$adj\ A = \begin{bmatrix} \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} \\ -\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} (0-1) & -(0-1) & (1-0) \\ -(0-1) & (0-1) & -(0-1) \\ (1-0) & -(0-1) & (0-1) \end{bmatrix}^T$$

$$= \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}^T$$

$$adj\ A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$A^{-1} = \frac{1}{|A|} adj\ A$

$$A^{-1} = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$A^2 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0+1+1 & 0+0+1 & 0+1+0 \\ 0+0+1 & 1+0+1 & 1+0+0 \\ 0+1+0 & 1+0+0 & 1+1+0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$3I = 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$A^2 - 3I = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$\frac{1}{2}(A^2 - 3I) = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

From (1) & (2), we get

$$A^{-1} = \frac{1}{2}(A^2 - 3I)$$

15. Decrypt the received encoded message $[2\ \ -3][20\ \ 4]$ with the encryption matrix $\begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$ and the decryption matrix as its inverse, where the system of codes are described by the numbers 1-26 to the letters $A - Z$ respectively, and the number 0 to a blank space. 3M

Let the encoding matrix be $A = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$

$|A| = -1 + 2 = 1 \neq 0 \quad \therefore A^{-1}$ exists

$adj\ A = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|} adj\ A$

$= \frac{1}{1}\begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$

$A^{-1} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$

$\therefore$ Decoding matrix $= A^{-1} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$

Coded row matrix	decoding matrix	decoded row matrix
$[2\ \ -3]$	$\begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$	$= [2+6\ \ 2+3] = [8\ \ 5]$
$[20\ \ 4]$	$\begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$	$= [20-8\ \ 20-4] = [12\ 16]$

So, the sequence of decoded row matrix = $[8\ \ 5][12\ 16]$

The message received = HELP

Thus, the message received is “HELP.”

EXERCISE 1.2

Concept Corner

Elementary Transformations of a matrix:

A matrix can be transformed to another matrix by certain operations called elementary row operations and elementary column operations.

Elementary Row and Column Operations:

Elementary row operations and elementary column operations on a matrix are known a **elementary transformations.**
we use the following notations for elementary row transformations.

(i) Interchanging of i^{th} and j^{th} rows is denoted by $R_i \leftrightarrow R_j$.

(ii) The multiplication of each element of i^{th} row by a non-zero constant λ is denoted by $R_i \rightarrow \lambda R_i$.

(iii) Addition to i^{th} row, a non-zero constant λ multiple of j^{th} row is denoted by $R_i \rightarrow R_i + \lambda R_j$.

Two matrices A and B of same order are said to be **equivalent** to one another if one can be obtained from the other by the applications of elementary transformations. Symbolically, we write $A \sim B$ to mean that the matrix A is equivalent to the matrix B.

Row-echelon form

A non-zero matrix E is said to be in a **row-echelon form** if.

(i) All zero rows of E occur below every non-zero row of E.

(ii) If the first non-zero element in any row i of E occurs in the j^{th} column of E, then all other entries in the j^{th} column of E below the first non-zero element of row i are zeros.

(iii) The first non-zero entry in the i^{th} row of E lies to the left of the first non- zero entry in $(i+1)^{th}$ row of E.

Note : A non-zero matrix is in a row-echelon form if all zero rows occur as bottom rows of the matrix, and if the first non-zero element in any lower row occurs to the right of the first non-zero entry in the higher row.

Rank of a matrix

The **rank of a matrix** A is defined as the order of a highest order non-vanishing minor of the matrix A. It is denoted by the symbol $\rho(A)$. The rank of a zero matrix is defined to be 0.

Note:

(i) If a matrix contains at-least one non-zero element, then $\rho(A) \geq 1$.

(ii) The rank of the identify matrix I_n is n.

(iii) If the rank of a matrix A is r, then there exists at-least one minor of A of order r which does not vanish and every minor of A of order $r+1$ and higher order (if any) vanishes.

(iv) If A is an $m \times n$ matrix, then $\rho(A) \leq \min\{m, n\}$ =minimum of m, n.

(v) A square matrix A of order n is invertible if and only if $\rho(A) = n$.

An elementary matrix is defined as a matrix which is obtained form an identity matrix by applying only one elementary transformation.

Theorem

Every non-singular matrix can be transformed to an identity matrix, by a sequence of elementary row operations.

Gauss-Jordan method

Transforming a non-singular matrix A to the form I_n by applying elementary row operations, is called **Gauss-Jordan method**. The steps in finding $A^{-1}$by Gauss-Jordan method are given below.

Step 1

Augment the identity matrix I_n on the right-side of A to get the matrix $[A|I_n]$.

Step 2

Obtain elementary matrices (row operations) $E_1, E_2, \ldots E_k$ such that $(E_k..E_2E_1)A = I_n$.
Apply $E_1, E_2 \ldots E_k$ on $[A|I_n]$. Then $[(E_k..E_2E_1)A|(E_k..E_2E_1)I_n]$. that is, $[I_n|A^{-1}]$.

1. Find the rank of the following matrices by minor method. (i) $\begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} -1 & 3 \\ 4 & -7 \\ 3 & -4 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & -2 & -1 & 0 \\ 3 & -6 & -3 & 1 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & -6 \\ 5 & 1 & -1 \end{bmatrix}$ (v) $\begin{bmatrix} 0 & 1 & 2 & 1 \\ 0 & 2 & 4 & 3 \\ 8 & 1 & 0 & 2 \end{bmatrix}$

(i) Let $A = \begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix}$ 2M

Order of the matrix A is 2×2

$\therefore \rho(A) \leq \min\{2,2\} = 2$

$|A| = 4 - 4 = 0$

$\therefore \rho(A) \neq 2$

Thus $\rho(A) = 1$

(ii) Let $A = \begin{bmatrix} -1 & 3 \\ 4 & -7 \\ 3 & -4 \end{bmatrix}$ 2M

order of the matrix A is 3×2

$\therefore \rho(A) \leq \min\{3,2\} = 2$

We find that the second order minor

$\begin{vmatrix} -1 & 3 \\ 4 & -7 \end{vmatrix} = 7 - 12 = -5 \neq 0.$

$\therefore \rho(A) = 2.$

(iii) Let $A = \begin{bmatrix} 1 & -2 & -1 & 0 \\ 3 & -6 & -3 & 1 \end{bmatrix}$ 2M PTA-5

order of the matrix A is 2×4

$\therefore \rho(A) \leq \min\{2,4\} = 2$

We find that the second order minor

$\begin{vmatrix} -1 & 0 \\ -3 & 1 \end{vmatrix} = -1 + 0 = -1 \neq 0.$

$\therefore \rho(A) = 2$

(iv) Let $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & -6 \\ 5 & 1 & -1 \end{bmatrix}$ 2M

Order of the matrix A is 3×3

$\therefore \rho(A) \leq \min\{3,3\} = 3$

There is only one third order minor of A

$|A| = 1(-4 + 6) + 2(-2 + 30) + 3(2 - 20)$

$= 1(2) + 2(28) + 3(-18)$

$= 2 + 56 - 54$

$= 4 \neq 0.$

$\therefore \rho(A) = 3$

(v) Let $A = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 0 & 2 & 4 & 3 \\ 8 & 1 & 0 & 2 \end{bmatrix}$ 2M

Order of the matrix A is 3×4

$\therefore \rho(A) \leq \min\{3,4\} = 3$

We find that the third order minor

$\begin{vmatrix} 0 & 1 & 1 \\ 0 & 2 & 3 \\ 8 & 1 & 2 \end{vmatrix} = 0 - 0 + 8(3 - 2)$

$= 8(1) = 8 \neq 0$

$\therefore \rho(A) = 3$

2. **Find the rank of the following matrices by row reduction method:**

(i) $\begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & -1 & 3 & 4 \\ 5 & -1 & 7 & 11 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 1 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} 3 & -8 & 5 & 2 \\ 2 & -5 & 1 & 4 \\ -1 & 2 & 3 & -2 \end{bmatrix}$

(i) Let $A = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & -1 & 3 & 4 \\ 5 & -1 & 7 & 11 \end{bmatrix}$ 3M

$\sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -3 & 1 & -2 \\ 0 & -6 & 2 & -4 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 5R_1 \end{matrix}$

$\sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -3 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_3 \to R_3 - 2R_2$

The last equivalent matrix is in row-echelon form. It has two non-zero rows.

$\therefore \rho(A) = 2$

(iii) Let $A = \begin{bmatrix} 3 & -8 & 5 & 2 \\ 2 & -5 & 1 & 4 \\ -1 & 2 & 3 & -2 \end{bmatrix}$ 3M

$\sim \begin{bmatrix} 3 & -8 & 5 & 2 \\ 0 & 1 & -7 & 8 \\ 0 & -2 & 14 & -4 \end{bmatrix} \begin{matrix} R_2 \to 3R_2 - 2R_1 \\ R_3 \to 3R_3 - R_1 \end{matrix}$

$\sim \begin{bmatrix} 3 & -8 & 5 & 2 \\ 0 & 1 & -7 & 8 \\ 0 & 0 & 0 & 12 \end{bmatrix} R_3 \to R_3 - 2R_2$

The last equivalent matrix is in row – echelon form. It has three non-zero rows.

$\therefore \rho(A) = 3$

(ii) Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 1 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix}$ 3M

$\sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & -4 & 4 \\ 0 & -3 & 2 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 - R_1 \end{matrix}$

$\sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & 0 & 8 \\ 0 & 0 & -4 \end{bmatrix} \begin{matrix} R_3 \to 7R_3 - 4R_2 \\ R_4 \to 4R_4 - 3R_3 \end{matrix}$

$\sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{bmatrix} R_4 \to 2R_4 + R_3$

The last equivalent matrix is in row-echelon form. It has three non-zero row.

$\therefore \rho(A) = 3$

3. **Find the inverse of each of the following by Gauss-Jordan method:**

(i) $\begin{bmatrix} 2 & -1 \\ 5 & -2 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 6 & -2 & -3 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$

(i) Let $A = \begin{bmatrix} 2 & -1 \\ 5 & -2 \end{bmatrix}$, Applying Gauss – Jordan method, we get 3M

$[A|I] = \left[\begin{array}{cc|cc} 2 & -1 & 1 & 0 \\ 5 & -2 & 0 & 1 \end{array}\right]$

$\to \left[\begin{array}{cc|cc} 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 5 & -2 & 0 & 1 \end{array}\right] R_1 \to \frac{1}{2}R_1$

$\to \left[\begin{array}{cc|cc} 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{5}{2} & 1 \end{array}\right] R_2 \to R_2 - 5R_1$

$\to \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & \frac{1}{2} & -\frac{5}{2} & 1 \end{array}\right] R_1 \to R_1 + R_2 \to \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & -5 & 2 \end{array}\right] R_2 \to 2R_2$

$\therefore A^{-1} = \begin{bmatrix} -2 & 1 \\ -5 & 2 \end{bmatrix}$

(ii) Let $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 6 & -2 & -3 \end{bmatrix}$ applying Gauss – Jordan method, we get 5M

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 6 & -2 & -3 & 0 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 4 & -3 & -6 & 0 & 1 \end{array}\right] \begin{array}{l} R_2 \rightarrow R_2 - R_1 \\ R_3 \rightarrow R_3 - 6R_1 \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & -4 & 1 \end{array}\right] R_3 \rightarrow R_3 - 4R_2$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & -3 & 1 \\ 0 & 0 & 1 & -2 & -4 & 1 \end{array}\right] R_2 \rightarrow R_2 + R_3$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -3 & 1 \\ 0 & 1 & 0 & -3 & -3 & 1 \\ 0 & 0 & 1 & -2 & -4 & 1 \end{array}\right] R_1 \rightarrow R_1 + R_2$$

$$\therefore A^{-1} = \begin{bmatrix} -2 & -3 & 1 \\ -3 & -3 & 1 \\ -2 & -4 & 1 \end{bmatrix}$$

(iii) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$ applying Gauss – Jordan method, we get 5M

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right] \begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - R_1 \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right] R_3 \rightarrow R_3 + 2R_2$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right] R_3 \rightarrow (-1)R_3$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right] \begin{array}{l} R_1 \rightarrow R_1 - 3R_3 \\ R_2 \rightarrow R_2 + 3R_3 \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right] R_1 \rightarrow R_1 - 2R_2$$

$$\therefore A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

EXERCISE 1.3

Concept Corner

- A system of linear equations having at least one solution is said to be **consistent.** A system of linear equations having no solution is said to be **inconsistent.**
- **Matrix Inversion Method**

  This method can be applied only when the coefficient matrix is a square matrix and non-singular.

  Consider the matrix equation $AX = B$, ...(1)

  Where A is a square matrix and non-singular. Since A is non-singular, A^{-1} exists and $A^{-1}A = AA^{-1} = I$.

  Pre-multiplying both sides of (1) by A^{-1}, we get

  $$A^{-1}(AX) = A^{-1}B.$$

  That is, $(A^{-1}A)X = A^{-1}B$.

  Hence, we get $X = A^{-1}B$.

1. Solve the following system of linear equations by matrix inversion method.

(i) $2x + 5y = -2, x + 2y = -3$ (ii) $2x - y = 8, 3x + 2y = -2$

(iii) $2x + 3y - y = 9, x + y + z = 9, 3x - y - z = -1$

(iv) $x + y + z - 2 = 0, 6x - 4y + 5z - 31 = 0, 5x + 2y + 2z = 13$

(i) $2x + 5y = -2, x + 2y = -3$ 3M

The matrix form of the system is $AX = B$, where

$$A = \begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, B = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$$

$|A| = 4 - 5 = -1 \neq 0$

$\therefore A^{-1}$ exists

$$adjA = \begin{bmatrix} 2 & -5 \\ -1 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} adjA$$

$$A^{-1} = \frac{1}{-1}\begin{bmatrix} 2 & -5 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 5 \\ 1 & -2 \end{bmatrix}$$

$AX = B$

$\Rightarrow X = A^{-1}B$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 & 5 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} -2 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 - 15 \\ -2 + 6 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -11 \\ 4 \end{bmatrix}$$

The solution is $(x = -11, y = 4)$

(ii) $2x - y = 8, 3x + 2y = -2$ 3M PTA-3

The matrix form of the system is $AX = B$ where

$$A = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, B = \begin{bmatrix} 8 \\ -2 \end{bmatrix}$$

$|A| = 4 + 3 = 7 \neq 0$

$\therefore A^{-1}$ exists

$$adjA = \begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} adjA = \frac{1}{7}\begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$$

$AX = B \Rightarrow X = A^{-1}B$

$$= \frac{1}{7}\begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 8 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 16 - 2 \\ -24 - 4 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 14 \\ -28 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$$

The solution is $(x = 2, y = -4)$

(iii) $2x+3y-z=9, x+y+z=9, 3x-y-z=-1$ 5M

The matrix form of the system is $AX=B$ where $A=\begin{bmatrix}2&3&-1\\1&1&1\\3&-1&-1\end{bmatrix}, X=\begin{bmatrix}x\\y\\z\end{bmatrix}, B=\begin{bmatrix}9\\9\\-1\end{bmatrix}$

$|A|=2[-1+1]-3[-1-3]-1[-1-3]=16\neq 0.$

$\therefore A^{-1}$ exists

$adjA=$

$$\begin{bmatrix}\begin{vmatrix}1&1\\-1&-1\end{vmatrix} & -\begin{vmatrix}1&1\\3&-1\end{vmatrix} & \begin{vmatrix}1&1\\3&-1\end{vmatrix}\\ -\begin{vmatrix}3&-1\\-1&-1\end{vmatrix} & \begin{vmatrix}2&-1\\3&-1\end{vmatrix} & -\begin{vmatrix}2&3\\3&-1\end{vmatrix}\\ \begin{vmatrix}3&-1\\1&1\end{vmatrix} & -\begin{vmatrix}2&-1\\1&1\end{vmatrix} & \begin{vmatrix}2&3\\1&1\end{vmatrix}\end{bmatrix}^T$$

$$=\begin{bmatrix}(-1+1)&-(-1-3)&(-1-3)\\-(-3-1)&(-2+3)&-(-2-9)\\(3+1)&-(2+1)&(2-3)\end{bmatrix}^T$$

$$=\begin{bmatrix}0&4&-4\\4&1&11\\4&-3&-1\end{bmatrix}^T$$

$$adjA=\begin{bmatrix}0&4&4\\4&1&-3\\-4&11&-1\end{bmatrix}$$

$$A^{-1}=\frac{1}{|A|}adjA$$

$$=\frac{1}{16}\begin{bmatrix}0&4&4\\4&1&-3\\-4&11&-1\end{bmatrix}$$

$AX=B\Rightarrow X=A^{-1}B$

$$\Rightarrow\begin{bmatrix}x\\y\\z\end{bmatrix}=\frac{1}{16}\begin{bmatrix}0&4&4\\4&1&-3\\-4&11&-1\end{bmatrix}\begin{bmatrix}9\\9\\-1\end{bmatrix}$$

$$=\frac{1}{16}\begin{bmatrix}0+36-4\\36+9+3\\-36+99+1\end{bmatrix}$$

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\frac{1}{16}\begin{bmatrix}32\\48\\64\end{bmatrix}=\begin{bmatrix}2\\3\\4\end{bmatrix}$$

$\therefore$ The solution is $(x=2, y=3, z=4)$

(iv) $x+y+z=2, 6x-4y+5z-31=0, 5x+2y+2z=13$

The matrix form of the system is $AX=B$, where $A=\begin{bmatrix}1&1&1\\6&-4&5\\5&2&2\end{bmatrix}, X=\begin{bmatrix}x\\y\\z\end{bmatrix}, B=\begin{bmatrix}2\\31\\13\end{bmatrix}$

$|A|=1(-8-10)-1(12-25)+1(12+20)=27\neq 0$

$\therefore A^{-1}$ exists. 5M

$$adjA=\begin{bmatrix}(-8-10)&-(12-25)&(12+20)\\-(2-2)&(2-5)&-(2-5)\\(5+4)&-(5-6)&(-4-6)\end{bmatrix}^T$$

$$=\begin{bmatrix}-18&13&32\\0&-3&3\\9&1&-10\end{bmatrix}^T=\begin{bmatrix}-18&0&9\\13&-3&1\\32&3&-10\end{bmatrix}$$

$$A^{-1}=\frac{1}{|A|}adjA=\frac{1}{27}\begin{bmatrix}-18&0&9\\13&-3&1\\32&3&-10\end{bmatrix}$$

$$AX=B\Rightarrow X=A^{-1}B=\frac{1}{27}\begin{bmatrix}-18&0&9\\13&-3&1\\32&3&-10\end{bmatrix}\begin{bmatrix}2\\31\\13\end{bmatrix}$$

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\frac{1}{27}\begin{bmatrix}-36+0+117\\26-93+13\\64+93-130\end{bmatrix}=\frac{1}{27}\begin{bmatrix}81\\-54\\27\end{bmatrix}$$

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}3\\-2\\1\end{bmatrix}$$ $\therefore$ The solution is $(x=3, y=-2, z=1)$

2. If $A = \begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$, find the products AB and BA and hence solve the system of equations $x + y + 2z = 1, 3x + 2y + z = 7, 2x + y + 3z = 2$. 5M

$$AB = \begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} -5+3+6 & -5+2+3 & -10+1+9 \\ 7+3-10 & 7+2-5 & 14+1-15 \\ 1-3+2 & 1-2+1 & 2-1+3 \end{bmatrix}$$
$$= \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$
$$= 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$\therefore AB = 4I_3$

$$BA = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} -5+7+2 & 1+1-2 & 3-5+2 \\ -15+14+1 & 3+2-1 & 9-10+1 \\ -10+7+3 & 2+1-3 & 6-5+3 \end{bmatrix}$$
$$= \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$
$$= 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$\therefore BA = 4I_3$

So, we get $AB = BA = 4I_3$

$\Rightarrow \left(\frac{1}{4}A\right)B = B\left(\frac{1}{4}A\right) = I_3$

Hence $B^{-1} = \frac{1}{4}A$

[$\because AB = BA = I \Rightarrow B^{-1} = A (or) A^{-1} = B$]

Matrix form of the given system of equations:

$$\begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix}$$
$$\Rightarrow B\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = B^{-1}\begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix}$$
$$= \frac{1}{4}A\begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix}$$
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -5+7+6 \\ 7+7-10 \\ 1-7+2 \end{bmatrix}$$
$$= \frac{1}{4}\begin{bmatrix} 8 \\ 4 \\ -4 \end{bmatrix}$$
$$= \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

$\therefore$ The solution is $(x = 2, y = 1, z = -1)$

3. A man is appointed in a job with a monthly salary of certain amount and a fixed amount of annual increment. If his salary was ₹19,800 per month at the end of the first month after 3 years of service and ₹23,400 per month at the end of the first month after 9 years of service, find his starting salary and his annual increment. (Use matrix inversion method to solve the problem.) 3M

Let the monthly salary = ₹ x

Annual increment = ₹ y

From the given information, we have

$$x + 3y = 19800$$
$$x + 9y = 23400$$

The matrix form is $AX = B$

where $A = \begin{bmatrix} 1 & 3 \\ 1 & 9 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 19800 \\ 23400 \end{bmatrix}$

$|A| = 9 - 3 = 6 \neq 0$

$\therefore A^{-1}$ exists

$$adjA = \begin{bmatrix} 9 & -3 \\ -1 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} adjA$$
$$= \frac{1}{6}\begin{bmatrix} 9 & -3 \\ -1 & 1 \end{bmatrix}$$

$AX = B \Rightarrow X = A^{-1}B$

$$= \frac{1}{6}\begin{bmatrix} 9 & -3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 19800 \\ 23400 \end{bmatrix}$$
$$X = \frac{1}{6}\begin{bmatrix} 178200 & -70200 \\ -19800 & +23400 \end{bmatrix}$$
$$X = \frac{1}{6}\begin{bmatrix} 108000 \\ 3600 \end{bmatrix}$$
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 18000 \\ 600 \end{bmatrix}$$

The solution is $x = 18000, y = 600$

$\therefore$ Monthly salary = ₹18,000 & Annual increment = ₹600

4. 4 men and 4 women can finish a piece of work jointly in 3 days while 2 men and 5 women can finish the same work jointly in 4 days. Find the time taken by one man alone and that of one woman alone to finish the same work by using matrix inversion method. 3M

Let

The number of days taken by a man to complete the work $= x$

The number of days taken by a woman to complete the work $= y$

From the given information, we have

$$\frac{4}{x} + \frac{4}{y} = \frac{1}{3}$$
$$\frac{2}{x} + \frac{5}{y} = \frac{1}{4}$$

The matrix form is $AX = B$ where

$$A = \begin{bmatrix} 4 & 4 \\ 2 & 5 \end{bmatrix}, X = \begin{bmatrix} \frac{1}{x} \\ \frac{1}{y} \end{bmatrix}, B = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{4} \end{bmatrix}$$

$|A| = 20 - 8 = 12 \neq 0 \therefore A^{-1}$ exists

$$A^{-1} = \frac{1}{|A|} adjA$$
$$= \frac{1}{12}\begin{bmatrix} 5 & -4 \\ -2 & 4 \end{bmatrix}$$

$AX = B \Rightarrow X = A^{-1}B$

$$X = \frac{1}{12}\begin{bmatrix} 5 & -4 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} \frac{1}{3} \\ \frac{1}{4} \end{bmatrix}$$
$$X = \frac{1}{12}\begin{bmatrix} \frac{5}{3} & -1 \\ -\frac{2}{3} & +1 \end{bmatrix} = \frac{1}{12}\begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$$
$$\begin{bmatrix} \frac{1}{x} \\ \frac{1}{y} \end{bmatrix} = \begin{bmatrix} \frac{1}{12} \times \frac{2}{3} \\ \frac{1}{12} \times \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{18} \\ \frac{1}{36} \end{bmatrix}$$
$$\frac{1}{x} = \frac{1}{18} \Rightarrow x = 18$$
$$\frac{1}{y} = \frac{1}{36} \Rightarrow y = 36$$

$\therefore$ Number of days taken by a man to complete the work = 18 days

Number of days taken by a woman to complete the work = 36 days

5. The prices of three commodities A, B and C are ₹ x, y and z per units respectively. A person P purchases 4 units of B and sells two units of A and 5 units of C. Person Q purchases 2 units of C and sells 3 units of A and one unit of B. Person R purchases one unit of A and sells 3 units of B and one unit of C. In the process, P,Q and R earn ₹ 15,000, ₹ 1,000 and ₹ 4,000 respectively. Find the prices per unit of A, B and C. (Use matrix inversion method to solve the problem.)

5M

Let the price per unit of

Commodity $A = ₹x$, Commodity $B = ₹y$, Commodity $C = ₹z$

From the given information, we get

$$2x - 4y + 5z = 15000$$
$$3x + y - 2z = 1000$$
$$-x + 3y + z = 4000$$

Note: 1. The amount separate on purchasing the commodity is negative
2. The amount earned by selling the commodity is positive.

The matrix from is $AX = B$ where

$$A = \begin{bmatrix} 2 & -4 & 5 \\ 3 & 1 & -2 \\ -1 & 3 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 15000 \\ 1000 \\ 4000 \end{bmatrix}$$

$|A| = 2(1+6) + 4(3-2) + 5(9+1) = 68 \neq 0$

$\therefore A^{-1}$ exists

$$adjA = \begin{bmatrix} (1+6) & -(3-2) & (9+1) \\ -(-4-15) & (2+5) & -(6-4) \\ (8-5) & -(-4-15) & (2+12) \end{bmatrix}^T$$

$$adjA = \begin{bmatrix} 7 & -1 & 10 \\ 19 & 7 & -2 \\ 3 & 19 & 14 \end{bmatrix}^T = \begin{bmatrix} 7 & 19 & 3 \\ -1 & 7 & 19 \\ 10 & -2 & 14 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} adjA = \frac{1}{68}\begin{bmatrix} 7 & 19 & 3 \\ -1 & 7 & 19 \\ 10 & -2 & 14 \end{bmatrix}$$

$$AX = B \Rightarrow X = A^{-1}B$$

$$X = \frac{1}{68}\begin{bmatrix} 7 & 19 & 3 \\ -1 & 7 & 19 \\ 10 & -2 & 14 \end{bmatrix}\begin{bmatrix} 15000 \\ 1000 \\ 4000 \end{bmatrix}$$

$$X = \frac{1}{68}\begin{bmatrix} 105000 + 19000 + 12000 \\ -15000 + 7000 + 76000 \\ 150000 - 2000 + 56000 \end{bmatrix}$$

$$X = \frac{1}{68}\begin{bmatrix} 136000 \\ 68000 \\ 204000 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2000 \\ 1000 \\ 3000 \end{bmatrix}$$

$\therefore$ The solution is $x = 2000, y = 1000, z = 3000$

Price per unit of $A =$ ₹ 2000 & Price per unit of $B =$ ₹ 1000 & Price per unit of $C =$ ₹3000

EXERCISE 1.4

Concept Corner

The Cramer's rule: $x_1 = \frac{\Delta_1}{\Delta},\ x_2 = \frac{\Delta_2}{\Delta},\ \ x_3 = \frac{\Delta_3}{\Delta}$

Where $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, $\Delta_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}$, $\Delta_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}$, $\Delta_3 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$

1. Solve the following systems of linear equations by Cramer's rule:

(i) $5x - 2y + 16 = 0, x + 3y - 7 = 0$ **(ii)** $\frac{3}{x} + 2y = 12, \frac{2}{x} + 3y = 13$

(iii) $3x + 3y - z = 11, 2x - y + 2z = 9, 4x + 3y + 2z = 25$

(iv) $\frac{3}{x} - \frac{4}{y} - \frac{2}{z} - 1 = 0, \frac{1}{x} + \frac{2}{y} + \frac{1}{z} - 2 = 0, \frac{2}{x} - \frac{5}{y} - \frac{4}{z} + 1 = 0$

(i) $5x - 2y = -16, x + 3y = 7$ **3M**

$\Delta = \begin{vmatrix} 5 & -2 \\ 1 & 3 \end{vmatrix} = 15 + 2 = 17$

$\Delta_1 = \begin{vmatrix} -16 & -2 \\ 7 & 3 \end{vmatrix} = -48 + 14 = -34$

$\Delta_2 = \begin{vmatrix} 5 & -16 \\ 1 & 7 \end{vmatrix} = 35 + 16 = 51$

By Cramer's rule, we get

$x = \frac{\Delta_1}{\Delta} = -\frac{34}{17} = -2$

$y = \frac{\Delta_2}{\Delta} = \frac{51}{17} = 3$

∴ The solution is $(x = -2, y = 3)$.

(ii) $\frac{3}{x} + 2y = 12, \frac{2}{x} + 3y = 13$ **3M**

$\Delta = \begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix} = 9 - 4 = 5$

$\Delta_1 = \begin{vmatrix} 12 & 2 \\ 13 & 3 \end{vmatrix} = 36 - 26 = 10$

$\Delta_2 = \begin{vmatrix} 3 & 12 \\ 2 & 13 \end{vmatrix} = 39 - 24 = 15$

$\frac{1}{x} = \frac{\Delta_1}{\Delta} = \frac{10}{5} = 2 \Rightarrow x = \frac{1}{2}$

$y = \frac{\Delta_2}{\Delta} = \frac{15}{5} = 3$

∴ The solution is $\left(x = \frac{1}{2}, y = 3\right)$

(iii) $3x + 3y - z = 11, 2x - y + 2z = 9, 4x + 3y + 2z = 25$ **5M**

$\Delta = \begin{vmatrix} 3 & 3 & -1 \\ 2 & -1 & 2 \\ 4 & 3 & 2 \end{vmatrix}$

$= 3[-2 - 6] - 3[4 - 8] - 1[6 + 4]$

$= -24 + 12 - 10 = -22$

$\Delta_1 = \begin{vmatrix} 11 & 3 & -1 \\ 9 & -1 & 2 \\ 25 & 3 & 2 \end{vmatrix}$

$= 11(-2 - 6) - 3(18 - 50) - 1(27 + 25)$

$= 11(-8) - 3(-32) - 1(52)$

$= -88 + 96 - 52 = -44$

$\Delta_2 = \begin{vmatrix} 3 & 11 & -1 \\ 2 & 9 & 2 \\ 4 & 25 & 2 \end{vmatrix}$

$= 3(18 - 50) - 11(4 - 8) - 1(50 - 36)$

$= 3(-32) - 11(-4) - 1(14)$

$= -96 + 44 - 14 = -66$

$\Delta_3 = \begin{vmatrix} 3 & 3 & 11 \\ 2 & -1 & 9 \\ 4 & 3 & 25 \end{vmatrix}$

$= 3(-25 - 27) - 3(50 - 36) + 11(6 + 4)$

$= 3(-52) - 3(14) + 11(10)$

$= -156 - 42 + 110 = -88$

By Cramer's Rule, we get

$x = \frac{\Delta_1}{\Delta} = \frac{-44}{-22} = 2$; $y = \frac{\Delta_2}{\Delta} = \frac{-66}{-22} = 3$; $z = \frac{\Delta_3}{\Delta} = \frac{-88}{-22} = 4$

∴ The solution is $(x = 2, y = 3, z = 4)$

(iv) $\frac{3}{x}-\frac{4}{y}-\frac{2}{z}=1$

$\frac{1}{x}+\frac{2}{y}+\frac{1}{z}=2$

$\frac{2}{x}-\frac{5}{y}-\frac{4}{z}=-1$

$$\Delta=\begin{vmatrix}3 & -4 & -2\\ 1 & 2 & 1\\ 2 & -5 & -4\end{vmatrix}$$

$= 3(-8+5)+4(-4-2)-2(-5-4)$

$= 3(-3)+4(-6)-2(-9)$

$= -9-24+18=-15$

$$\Delta_1=\begin{vmatrix}1 & -4 & -2\\ 2 & 2 & 1\\ -1 & -5 & -4\end{vmatrix}$$

$= 1(-8+5)+4(-8+1)-2(-10+2)$

$= 1(-3)+4(-7)-2(-8)$

$= -3-28+16=-15$

5M

$$\Delta_2=\begin{vmatrix}3 & 1 & -2\\ 1 & 2 & 1\\ 2 & -1 & -4\end{vmatrix}$$

$= 3(-8+1)-1(-4-2)-2(-1-4)$

$= 3(-7)-1(-6)-2(-5)$

$= -21+6+10=-5$

$\Delta_2=-5$

$$\Delta_3=\begin{vmatrix}3 & -4 & 1\\ 1 & 2 & 2\\ 2 & -5 & -1\end{vmatrix}$$

$= 3(-2+10)+4(-1-4)+1(-5-4)$

$= 3(8)+4(-5)+1(-9)$

$= 24-20-9=-5$

$\Delta_3=-5$

$\frac{1}{x}=\frac{\Delta_1}{\Delta}=\frac{-15}{-15}=1 \Rightarrow x=1$

$\frac{1}{y}=\frac{\Delta_2}{\Delta}=-\frac{5}{-15}=\frac{1}{3} \Rightarrow y=3$

$\frac{1}{z}=\frac{\Delta_3}{\Delta}=-\frac{5}{-15}=\frac{1}{3} \Rightarrow z=3$

$\therefore$ The solution is $(x=1, y=3, z=3)$

2. In a competitive examination, one mark is awarded for every correct answer while $\frac{1}{4}$ mark is deducted for every wrong answer. A student answered 100 questions and got 80 marks. How many questions did he answer correctly? (Use Cramer's rule to solve the problem). 3M

Let the number of question Answered correctly $= x$

The number of question Answered wrongly $= y$

$\therefore x+y=100$ ………………(1)

Marks awarded for one

Correct answer $= 1$

Wrong answer $= -\frac{1}{4}$

$\therefore (1\times x)+\left(-\frac{1}{4}\times y\right)=80$

Multiplying by 4, we get

$4x-y=320$ ……………..(2)

$x+y=100$

$4x-y=320$

$\Delta=\begin{vmatrix}1 & 1\\ 4 & -1\end{vmatrix}=-1-4=-5$

$\Delta_1=\begin{vmatrix}100 & 1\\ 320 & -1\end{vmatrix}=-100-320=-420$ $\quad\Delta_1=-420$

$\Delta_2=\begin{vmatrix}1 & 100\\ 4 & 320\end{vmatrix}=320-400=-80$ $\quad\Delta_2=-80$

$x=\frac{\Delta_1}{\Delta}=\frac{-420}{-5}=84$ & $y=\frac{\Delta_2}{\Delta}=\frac{-80}{-5}=16$ $\therefore$ The student answered 84 questions correctly.

3. A chemist has one solution which is 50% acid and another which is 25% acid. How much each should be mixed to make 10 litres of a 40% acid solution? (Use Cramer's rule to solve the problem). 3M

Let x and y be the amount of solution containing 50% and 25% acid respectively.

From the given data,

$$x + y = 10 \quad \ldots\ldots\ldots\ldots(1)$$

50% of x +25% of $y = 40\%$ of 10

$$\frac{50}{100}x + \frac{25}{100}y = \frac{40}{100}(10)$$

$$50x + 25y = 400 \quad \ldots\ldots\ldots\ldots(2)$$

$$\Delta = \begin{vmatrix} 1 & 1 \\ 50 & 25 \end{vmatrix} = 25 - 50 = -25$$

$$\Delta_x = \begin{vmatrix} 10 & 1 \\ 400 & 25 \end{vmatrix} = 250 - 400 = -150$$

$$\Delta_y = \begin{vmatrix} 1 & 100 \\ 50 & 400 \end{vmatrix}$$

$$= 400 - 500 = -100$$

$$x = \frac{\Delta_x}{\Delta} = \frac{-150}{-25} = 6$$

$$y = \frac{\Delta_y}{\Delta} = \frac{-100}{-5} = 4$$

6 litres of solution containing 50% acid and 4 litres of solution containing 25% acid must be mixed to make 40% acid solution

4. A fish tank can be filled in 10 minutes using both pumps A and B simultaneously. However, pump B can pump water in or out at the same rate. If pump B is inadvertently run in reverse, then the tank will be filled in 30 minutes. How long would it take each pump to fill the tank by itself? (use Cramer's rule to solve the problem). 3M

Let

The time taken by pump A to fill the tank by itself $= x$ minutes

The time taken by pump B to fill the tank by itself $= y$ minutes

So, the part of the tank filled by pump A in 1 minute $= \frac{1}{x}$

The part of the tank filled by pump B in 1 minute $= \frac{1}{y}$

The part of the tank filled by both pumps A & B in 1 minute $= \frac{1}{10}$

$$\therefore \frac{1}{x} + \frac{1}{y} = \frac{1}{10}$$

If pump B runs in reverse, then the tank will be filled by both pumps in 30 minutes.

In this case, the part of the tank filled by both pumps A & B in 1 minute $= \frac{1}{30}$

$$\therefore \frac{1}{x} - \frac{1}{y} = \frac{1}{30}$$

Let $a = \frac{1}{x}$ and $b = \frac{1}{y}$

$$a + b = \frac{1}{10} \Rightarrow 10a + 10b = 1$$

$$a - b = \frac{1}{30} \Rightarrow 30a - 30b = 1$$

$$\Delta = \begin{vmatrix} 10 & 10 \\ 30 & -30 \end{vmatrix} = -300 - 300 = -600$$

$$\Delta_1 = \begin{vmatrix} 1 & 10 \\ 1 & -30 \end{vmatrix} = -30 - 10 = -40$$

$$\Delta_2 = \begin{vmatrix} 10 & 1 \\ 30 & 1 \end{vmatrix} = 10 - 30 = -20$$

$$a = \frac{\Delta_1}{\Delta} = \frac{-40}{-600} = \frac{1}{15}$$

$$b = \frac{\Delta_2}{\Delta} = \frac{-20}{-600} = +\frac{1}{30}$$

$$a = \frac{1}{x} = \frac{1}{15} \Rightarrow x = 15$$

$$b = \frac{1}{y} = \frac{1}{30} \Rightarrow y = 30.$$

Pump A will take 15 minutes to fill the tank by itself.

Pump B will take 30 minutes to fill the tank by itself.

5. A family of 3 people went out for dinner in a restaurant. The cost of two dosai, three idlies and two vadais is ₹150. The cost of the two dosai, two idlies and four vadais is ₹200. The cost of five dosai, four idlies and two vadais is ₹250. The family has ₹350 in hand and they ate 3 dosai and six idlies and six vadais. Will they be able to manage to pay the bill within the amount they had? 5M

Let The cost of one dosai $= ₹\ x$

The cost of one idly $= ₹\ y$

The cost of one vadai $= ₹z$

According to the given information, we get

$2x + 3y + 2z = 150$

$2x + 2y + 4z = 200$

$5x + 4y + 2z = 250$

$$\Delta = \begin{vmatrix} 2 & 3 & 2 \\ 2 & 2 & 4 \\ 5 & 4 & 2 \end{vmatrix}$$

$= 2(4-16) - 3(4-20) + 2(8-10)$

$= 2(-12) - 3(-16) + 2(-2)$

$= -24 + 48 - 4 = 20$

$\Delta = 20$

$$\Delta_1 = \begin{vmatrix} 150 & 3 & 2 \\ 200 & 2 & 4 \\ 250 & 4 & 2 \end{vmatrix}$$

$= 150(4-16) - 3(400-1000) + 2(800-500)$

$= 150(-12) - 3(-600) + 2(300)$

$= -1800 + 1800 + 600 = 600$

$\therefore \Delta_1 = 600$

$$\Delta_2 = \begin{vmatrix} 2 & 150 & 2 \\ 2 & 200 & 4 \\ 5 & 250 & 2 \end{vmatrix}$$

$= 2(400-1000) - 150(4-20) + 2(500-1000)$

$= 2(-600) - 150(-16) + 2(-500)$

$= -1200 + 2400 - 1000 = 200$

$\therefore \Delta_2 = 200$

$$\Delta_3 = \begin{vmatrix} 2 & 3 & 150 \\ 2 & 2 & 200 \\ 5 & 4 & 250 \end{vmatrix}$$

$= 2(500-800) - 3(500-1000) + 150(8-10)$

$= 2(-300) - 3(-500) + 150(-2)$

$= -600 + 1500 - 300 = 600$

$\Delta_3 = 600$

By Cramer's Rule, we get

$$x = \frac{\Delta_1}{\Delta} = \frac{600}{20} = 30$$

$$y = \frac{\Delta_2}{\Delta} = \frac{200}{20} = 10$$

$$z = \frac{\Delta_3}{\Delta} = \frac{600}{20} = 30$$

$\therefore$ The cost of one dosai $= ₹30$

The cost of one idly $= ₹10$

The cost of one vadai $= ₹30$

The cost of 3 dosai and six idly and six vadai $= 3(30) + 6(10) + 6(30)$

$= 90 + 60 + 180 = ₹\ 330$

Since the family has ₹ 350 in hand, they will be able to manage to pay the bill.

EXERCISE 1.5

1. Solve the following systems of linear equations by Gaussian elimination method.

(i) $2x-2y+3z=2, x+2y-z=3,$
$3x-y+2z=1$ 5M

The augmented matrix is

$$[A|B]=\left[\begin{array}{ccc|c}2&-2&3&2\\1&2&-1&3\\3&-1&2&1\end{array}\right]$$

$$\to\left[\begin{array}{ccc|c}1&2&-1&3\\2&-2&3&2\\3&-1&2&1\end{array}\right] R_1\leftrightarrow R_2$$

$$\to\left[\begin{array}{ccc|c}1&2&-1&3\\0&-6&5&-4\\0&-7&5&-8\end{array}\right] \begin{array}{l}R_2\to R_2-2R_1\\R_3\to R_3-3R_1\end{array}$$

$$\to\left[\begin{array}{ccc|c}1&2&-1&3\\0&-6&5&-4\\0&0&-5&-20\end{array}\right] R_3\to 6R_3-7R_2$$

Writing the equivalent system of equation from the row-echelon matrix, we get

$x+2y-z=3$……………..(1)

$-6y+5z=-4$……………..(2)

$-5z=-20$……………………(3)

(3)$\Rightarrow z=\frac{20}{5}=4\Rightarrow z=4$

Substituting $z=4$ in (2), we get

$-6y+5(4)=-4$

$-6y=-4-20=-24$

$y=\frac{-24}{-6}=4$

$y=4$

Substituting $y=4$ and $z=4$ in (1), we get

$x+2(4)-4=3$

$x+4=3$

$x=3-4=-1$

$x=-1$

$\therefore$ The solution is $(x=-1, y=4, z=4)$

(ii) $2x+4y+6z=22,$
$3x+8y+5z=27, -x+y+2z=2$

The augmented matrix is 5M

$$[A|B]=\left[\begin{array}{ccc|c}2&4&6&22\\3&8&5&27\\-1&1&2&2\end{array}\right]$$

$$\to\left[\begin{array}{ccc|c}1&2&3&11\\3&8&5&27\\-1&1&2&2\end{array}\right] R_1\to\frac{1}{2}R_1$$

$$\to\left[\begin{array}{ccc|c}1&2&3&11\\0&2&-4&-6\\0&3&5&13\end{array}\right] \begin{array}{l}R_2\to R_2-3R_1\\R_3\to R_3+R_1\end{array}$$

$$\to\left[\begin{array}{ccc|c}1&2&3&11\\0&1&-2&-3\\0&3&5&13\end{array}\right] R_2\to\frac{1}{2}R_2$$

$$\to\left[\begin{array}{ccc|c}1&2&3&11\\0&1&-2&-3\\0&0&11&22\end{array}\right] R_3\to R_3-3R_2$$

Writing the equivalent system of equations from the row – echelon matrix, we get

$x+2y+3z=11$…………(1)

$y-2z=-3$………………(2)

$11z=22$………………………(3)

(3) $\Rightarrow z=\frac{22}{11}=2$

$z=2$

Substituting $z=2$ in (2), we get

$y-2(2)=-3$

$y-4=-3$

$y=-3+4=1$

$y=1$

Substituting $y=1$ & $z=2$ in (1), we get

$x+2(1)+3(2)=11$

$x+2+6=11\Rightarrow x=11-8=3$

$x=3$

$\therefore$ The solution is $(x=3, y=1, z=2)$

2. **If $ax^2 + bx + c$ is divided by $x + 3, x - 5$, and $x - 1$, the remainders are 21, 61 and 9 respectively. Find a, b and c. (Use Gaussian elimination method.)** PTA-3 5M

Let $P(x) = ax^2 + bx + c$

Given: $P(x) \div (x + 3)$ and leaves the remainder 21

$\therefore P(-3) = a(-3)^2 + b(-3) + c = 21$

$9a - 3b + c = 21$

Given: $P(x) \div (x - 5)$ and leaves the remainder 61

$\therefore P(5) = a(5)^2 + b(5) + c = 61$

$25a + 5b + c = 61$

Given: $P(x) \div (x - 1)$ and leave the remainder 9

$\therefore P(+1) = a(+1)^2 + b(+1) + c = 9$

$a + b + c = 9$

$\therefore$ The system of linear equations:

$9a - 3b + c = 21$

$25a + 5b + c = 61$

$a + b + c = 9$

The augmented matrix is $[A|B] = \left[\begin{array}{ccc|c} 9 & -3 & 1 & 21 \\ 25 & 5 & 1 & 61 \\ 1 & 1 & 1 & 9 \end{array}\right]$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 25 & 5 & 1 & 61 \\ 9 & -3 & 1 & 21 \end{array}\right] R_1 \leftrightarrow R_3$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -20 & -24 & -164 \\ 0 & -12 & -8 & -60 \end{array}\right] \begin{array}{l} R_2 \rightarrow R_2 - 25R_1 \\ R_3 \rightarrow R_3 - 9R_1 \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -5 & -6 & -41 \\ 0 & -3 & -2 & -15 \end{array}\right] \begin{array}{l} R_2 \rightarrow \frac{R_2}{4} \\ R_3 \rightarrow \frac{R_3}{4} \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -5 & -6 & -41 \\ 0 & 0 & 8 & 48 \end{array}\right] R_3 \rightarrow 5R_3 - 3R_2$$

Writing the equivalent equation from the row – echelon matrix, we get

$a + b + c = 9$………………(1)

$-5b - 6c = -41$…………..(2)

$8c = 48$…………………………(3)

$(3) \Rightarrow c = \frac{48}{8} = 6$

$c = 6$

Substituting $c = 6$ in (2), we get

$-5b - 6(6) = -41$

$-5b - 36 = -41$

$-5b = -41 + 36 = -5$

$b = \frac{-5}{-5} = 1$

$b = 1$

Substituting $b = 1, c = 6$ in (1), we get

$a + 1 + 6 = 9$

$a = 9 - 7 = 2$

$a = 2$

$\therefore$ The solution is $(a = 2, b = 1, c = 6)$

3. An amount of ₹65,000 is invested in three bonds at the rates of 6%, 8% and 9% per annum respectively. The total annual income is ₹4800. The income from the third bond is ₹600 more than that from the second bond. Determine the price of each bond. (Use Gaussian elimination method.)

5M

Let the price of 6%, 8%, 9% bond be ₹x, ₹y, ₹z respectively.

From the given data

$$x + y + z = 65000 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$(6\%\ x) + (8\%\ y) + (9\%\ z) = 4800$

$$6x + 8y + 9z = 480000 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$9\%\ z = 600 + 8\%\ y$$

$$-8y + 9z = 60000 \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

From (1), (2) and (3)

The augmented matrix is

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 65000 \\ 6 & 8 & 9 & 480000 \\ 0 & -8 & 9 & 60000 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 65000 \\ 0 & 2 & 3 & 90000 \\ 0 & -8 & 9 & 60000 \end{array}\right] R_2 \rightarrow R_2 - 6R_1$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 65000 \\ 0 & 2 & 3 & 90000 \\ 0 & 0 & 21 & 42000 \end{array}\right] R_3 \rightarrow R_3 + 4R_2$$

$x + y + z = 65000 \quad \ldots\ldots\ldots\ldots\ldots (4)$

$2y + 3z = 90000 \quad \ldots\ldots\ldots\ldots\ldots (5)$

$21z = 42000$

$z = \frac{42000}{21} = 20{,}000$

$(5) \Rightarrow 2y + 3(20000) = 90000$

$2y = 90000 - 60000$

$2y = 30000$

$y = 15000$

$(4) \Rightarrow x + 15000 + 20000 = 65000$

$x = 65000 - 35000$

$x = 30{,}000$

$\therefore$ The price of 6% bond = ₹ 30,000

The price of 8%bond = ₹ 15,000

The price of 9% bond = ₹20,000

4. A boy is walking along the path $y = ax^2 + bx + c$ through the points $(-6, 8), (-2, -12)$, and $(3, 8)$. He wants to meet his friend at $P(7, 60)$. Will he meet his friend? (Use Gaussian elimination method.) 5M

The path $y = ax^2 + bx + c$ passes through the points $(-6,8), (-2,-12)$ and $(3,8)$

So, we get the system of equations

$$8 = a(-6)^2 + b(-6) + c \Rightarrow 36a - 6b + c = 8$$
$$-12 = a(-2)^2 + b(-2) + c \Rightarrow 4a - 2b + c = -12$$
$$8 = a(3)^2 + b(3) + c \Rightarrow 9a + 3b + c = 8$$

The augmented matrix is

$$[A|B] = \left[\begin{array}{ccc|c} 36 & -6 & 1 & 8 \\ 4 & -2 & 1 & -12 \\ 9 & 3 & 1 & 8 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 36 & -6 & 1 & 8 \\ 0 & -12 & 8 & -116 \\ 0 & 18 & 3 & 24 \end{array}\right] \begin{array}{l} R_2 \rightarrow 9R_2 - R_1 \\ R_3 \rightarrow 4R_3 - R_1 \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|c} 36 & -6 & 1 & 8 \\ 0 & -3 & 2 & -29 \\ 0 & 6 & 1 & 8 \end{array}\right] \begin{array}{l} R_2 \rightarrow \frac{1}{4}R_2 \\ R_3 \rightarrow \frac{1}{3}R_3 \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|c} 36 & -6 & 1 & 8 \\ 0 & -3 & 2 & -29 \\ 0 & 0 & 5 & -50 \end{array}\right] R_3 \rightarrow R_3 + 2R_2$$

Writing the equivalent equations from the row-echelon matrix, we get

$$36a - 6b + c = 8 \quad \ldots\ldots\ldots\ldots\ldots (1)$$
$$-3b + 2c = -29 \quad \ldots\ldots\ldots\ldots\ldots (2)$$
$$5c = -50 \quad \ldots\ldots\ldots\ldots\ldots (3)$$

$(3) \Rightarrow c = -\frac{50}{5} = -10$

$(2) \Rightarrow -3b + 2(-10) = -29$

$-3b = -29 + 20 = -9$

$b = \frac{-9}{-3} = 3 \qquad \Rightarrow b = 3$

$(1) \Rightarrow 36a - 6(3) - 10 = 8$

$36a - 28 = 8$

$36a = 8 + 28 = 36$

$a = \frac{36}{36} = 1$

$\therefore$ Equation of the path is $y = x^2 + 3x - 10$

Substituting $x = 7$, we get

$y = 7^2 + 3(7) - 10$

$= 49 + 21 - 10 = 60$

Thus, the path passes through the point $P(7,60)$.

Hence, the boy will meet his friend.

EXERCISE 1.6

Concept Corner

Rouche'-Capelli Theorem:

A system of linear equations, written in the form as $AX = B$, is consistent if any only if the rank of the coefficient matrix is equal to the rank of the augmented matrix; that is, $\rho(A) = \rho([A|B])$.

System of Non-homogeneous Equation

⇓

$\rho(A) = \rho([A\|B]) =$ number of unknowns	$\rho(A) = \rho([A\|B]) <$ number of unknowns	$\rho(A) \neq \rho([A\|B])$
⇓	⇓	⇓
The system is consistent and has unique solution.	The system is consistent and has infinite number of solution.	The system is inconsistent and has no solution.

1. **Test for consistency and if possible, solve the following systems of equations by rank method.**
   (i) $x - y + 2z = 2, 2x + y + 4z = 7, 4x - y + z = 4$ 5M

The matrix form of the system is $AX = B$, where

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 4 \\ 4 & -1 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 2 \\ 7 \\ 4 \end{bmatrix}$$

The augmented matrix is

$$[A|B] = \left[\begin{array}{ccc|c} 1 & -1 & 2 & 2 \\ 2 & 1 & 4 & 7 \\ 4 & -1 & 1 & 4 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 2 & 2 \\ 0 & 3 & 0 & 3 \\ 0 & 3 & -7 & -4 \end{array}\right] \begin{matrix} \\ R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - 4R_1 \end{matrix}$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 2 & 2 \\ 0 & 3 & 0 & 3 \\ 0 & 0 & -7 & -7 \end{array}\right] R_3 \rightarrow R_3 - R_2$$

The last equivalent matrix is in row-echelon form and has three non-zero rows.

$\therefore \rho[A|B] = 3$

$$\text{Also } A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 0 \\ 0 & 0 & -7 \end{bmatrix}$$

It is also in the echelon form and it has also three non-zero rows. $\therefore \rho(A) = 3$

Since $\rho(A) = \rho[A|B] = 3$ =no.of unknowns, the given system is consistent and has a unique solution.

The equivalent system of equations:

$x - y + 2z = 2$ .... (1)

$3y = 3$ .... (2)

$-7z = -7$ .... (3)

(3) $\Rightarrow -7z = -7$

$z = \frac{-7}{-7} = 1$

$z = 1$

(2) $\Rightarrow 3y = 3$

$y = \frac{3}{3} = 1$

$y = 1$

Substituting, $y = 1, z = 1$ in (1), we get

$x - 1 + 2 = 2$

$x + 1 = 2 \Rightarrow x = 2 - 1 = 1$

$\Rightarrow x = 1$

$\therefore$ The solution is $(x = 1, y = 1, z = 1)$

(ii) $3x + y + z = 2, x - 3y + 2z = 1, 7x - y + 4z = 5$ 5M

The matrix form of the system is $AX = B$, where

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & -3 & 2 \\ 7 & -1 & 4 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$$

The augmented matrix is

$$[A, B] = \left[\begin{array}{ccc|c} 3 & 1 & 1 & 2 \\ 1 & -3 & 2 & 1 \\ 7 & -1 & 4 & 5 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & -3 & 2 & 1 \\ 3 & 1 & 1 & 2 \\ 7 & -1 & 4 & 5 \end{array}\right] R_1 \leftrightarrow R_2$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & -3 & 2 & 1 \\ 0 & 10 & -5 & -1 \\ 0 & 20 & -10 & -2 \end{array}\right] \begin{array}{l} \\ R_2 \rightarrow R_2 - 3R_1 \\ R_3 \rightarrow R_3 - 7R_1 \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & -3 & 2 & 1 \\ 0 & 10 & -5 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] R_3 \rightarrow R_3 - 2R_1$$

The last equivalent matrix is in row-echelon form and has two non-zero rows.

$\therefore \rho[A|B] = 2$

Also $A = \begin{bmatrix} 1 & -3 & 2 \\ 0 & 10 & -5 \\ 0 & 0 & 0 \end{bmatrix}$

It has two non-zero rows $\therefore \rho(A) = 2$

Since $\rho(A) = \rho[A|B] = 2 <$ Number of unknowns, the given system is consistent and has infinitely many solutions.

The equivalent system of equations:

$x - 3y + 2z = 1$ .... (1)

$10y - 5z = -1$ .... (2)

Let $z = t, t \in R$

$(2) \Rightarrow 10y - 5t = -1$

$\Rightarrow 10y = 5t - 1 \Rightarrow y = \frac{5t-1}{10}$

$(1) \Rightarrow x - \frac{3}{10}(5t - 1) + 2t = 1$

$x + \frac{-15t+3+20t}{10} = 1$

$x + \frac{5t+3}{10} = 1$

$x = \frac{10-5t-3}{10} = \frac{7-5t}{10}$

$\therefore$ The solution is $\left(x = \frac{1}{10}(7 - 5t),\ y = \frac{1}{10}(5t - 1), z = t\right), t \in R$

(iii) $2x+2y+z=5, x-y+z=1, 3x+y+2z=4$ 5M

The matrix form of the system is $AX=B$, where

$$A=\begin{bmatrix}2&2&1\\1&-1&1\\3&1&2\end{bmatrix}, X=\begin{bmatrix}x\\y\\z\end{bmatrix}, B=\begin{bmatrix}5\\1\\4\end{bmatrix}$$

The augmented matrix is

$$[A,B]=\left[\begin{array}{ccc|c}2&2&1&5\\1&-1&1&1\\3&1&2&4\end{array}\right]$$

$$\rightarrow\left[\begin{array}{ccc|c}1&-1&1&1\\2&2&1&5\\3&1&2&4\end{array}\right] R_1\leftrightarrow R_2$$

$$\rightarrow\left[\begin{array}{ccc|c}1&-1&1&1\\0&4&-1&3\\0&4&-1&1\end{array}\right] \begin{array}{l}R_2\rightarrow R_2-2R_1\\R_3\rightarrow R_3-3R_1\end{array}$$

$$\rightarrow\left[\begin{array}{ccc|c}1&-1&1&1\\0&4&-1&3\\0&0&0&-2\end{array}\right] R_3\rightarrow R_3-R_1$$

The last equivalent matrix is in row-echelon form and it has three non-zero rows.$\therefore\ \rho[A|B]=3$

$$\text{Also } A=\begin{bmatrix}1&-1&1\\0&4&-1\\0&0&0\end{bmatrix}$$

It is in row – echelon form and it has two non-zero rows $\therefore\ \rho(A)=2$

Since $\rho(A)\neq\rho[A|B]$ the given system is inconsistent and has no solutions.

(iv) $2x-y+z=2, 6x-3y+3z=6, 4x-2y+2z=4$ PTA-5 5M

The matrix form of the system is $AX=B$, where

$$A=\begin{bmatrix}2&-1&1\\6&-3&3\\4&-2&2\end{bmatrix}, X=\begin{bmatrix}x\\y\\z\end{bmatrix}, B=\begin{bmatrix}2\\6\\4\end{bmatrix}$$

The augmented matrix is

$$[A|B]=\left[\begin{array}{ccc|c}2&-1&1&2\\6&-3&3&6\\4&-2&2&4\end{array}\right]$$

$$\rightarrow\left[\begin{array}{ccc|c}2&-1&1&2\\0&0&0&0\\0&0&0&0\end{array}\right] \begin{array}{l}R_2\rightarrow R_2-3R_1\\R_3\rightarrow R_3-2R_1\end{array}$$

The last equivalent matrix is in row-echelon form and has one non-zero rows.

$\therefore\ \rho[A|B]=1$

$$\text{Also } A=\begin{bmatrix}2&-1&1\\0&0&0\\0&0&0\end{bmatrix}$$

A is in row – echelon form and it has one non-zero row.

$\therefore\ \rho(A)=1$

Since $\rho(A)=\rho[A|B]=1$ <No.of unknowns, the given system is consistent and it has infinitely many solutions.

The equivalent system of equations:

$2x-y+z=2$

Let $y=s$ and $z=t$ where $s,t\in R$

$2x-s+t=2$

$2x=2+s-t$

$x=\frac{1}{2}(2+s-t)$

$\therefore$ The solution is $x=\frac{1}{2}(2+s-t)$,

$y=s,\ z=t\ , s,t\in R$

2. Find the value of k for which the equations $kx - 2y + z = 1, x - 2ky + z = -2$, $x - 2y + kz = 1$ have 5M

(i) no solution **(ii) unique solution** **(iii) infinitely many solution**

$kx - 2y + z = 1, x - 2ky + z = -2, x - 2y + kz = 1$

The matrix form of the system is $AX = B$, where

$$A = \begin{bmatrix} k & -2 & 1 \\ 1 & -2k & 1 \\ 1 & -2 & k \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

The augmented matrix is

$$[A|B] = \left[\begin{array}{ccc|c} k & -2 & 1 & 1 \\ 1 & -2k & 1 & -2 \\ 1 & -2 & k & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & -2 & k & 1 \\ 1 & -2k & 1 & -2 \\ k & -2 & 1 & 1 \end{array}\right] R_1 \leftrightarrow R_3$$

$$\to \left[\begin{array}{ccc|c} 1 & -2 & k & 1 \\ 0 & 2-2k & 1-k & -3 \\ 0 & 2k-2 & 1-k^2 & 1-k \end{array}\right] \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - kR_1 \end{matrix}$$

$$\to \left[\begin{array}{ccc|c} 1 & -2 & k & 1 \\ 0 & 2-2k & 1-k & -3 \\ 0 & 0 & 2-k-k^2 & -2-k \end{array}\right] R_3 \to R_3 + R_2$$

$$\to \left[\begin{array}{ccc|c} 1 & -2 & k & 1 \\ 0 & 2(1-k) & 1-k & -3 \\ 0 & 0 & (2+k)(1-k) & -(2+k) \end{array}\right]$$

Case (i) If $k = 1$

then $\rho[A|B] = 3$ and$\rho(A) = 1$

Since $\rho(A) \neq \rho[A|B]$, the given system of equations is inconsistent and has no solution.

Case (ii) If $k \neq -d2, k \neq 1$,

then $\rho[A|B] = 3$ and $\rho[A] = 3$

Since $\rho[A] = \rho[A|B] = 3 =$ number of unknowns, the given system is consistent and has a unique solution

Case (iii) If $k = -2$,

then $\rho[A|B] = 2$ and $\rho[A] = 2$

Since $\rho[A|B] = \rho(A) = 2 <$ number of unknowns, the given system of equations is consistent and has infinitely many solution.

The given system has

(i) no solution when $k = 1$

(ii) unique solution when $k \neq 1, k \neq -2$

(iii) infinitely many solution when $k = -2$

3. Investigate the values of λ and μ the system of linear equations $2x+3y+5z=9$, $7x+3y-5z=8, 2x+3y+\lambda z=\mu$, have (i) no solution (ii) a unique solution (iii) an infinite number of solutions. 5M

$2x+3y+5z=9, 7x+3y-5z=8, 2x+3y+\lambda z=\mu$

The matrix form of the system is $AX=B$, where

$$A=\begin{bmatrix}2&3&5\\7&3&-5\\2&3&\lambda\end{bmatrix}, X=\begin{bmatrix}x\\y\\z\end{bmatrix}, B=\begin{bmatrix}9\\8\\\mu\end{bmatrix}$$

The augmented matrix is

$$[A|B]=\left[\begin{array}{ccc|c}2&3&5&9\\7&3&-5&8\\2&3&\lambda&\mu\end{array}\right]\rightarrow\left[\begin{array}{ccc|c}2&3&5&9\\0&-15&-45&-47\\0&0&\lambda-5&\mu-9\end{array}\right]\begin{array}{l}R_2\rightarrow 2R_2-7R_1\\R_3\rightarrow R_3-R_1\end{array}$$

Case (i)

If $\lambda=5$ and $\mu\neq 9$, then $\rho(A)=2$ and $\rho[A|B]=3$

Since $\rho(A)\neq\rho[A|B]$, the given system of equations is inconsistent and has no solution.

Case (ii)

If $\lambda\neq 5$ and $\mu\in R$, then

Since $\rho(A)=\rho[A|B]=3=$ no. of. unknowns, the given system of equations is consistent and has a unique solution.

Case (iii)

If $\lambda=5$ and $\mu=9$ then $\rho(A)=2$ and $\rho[A|B]=2<$ No. of unknowns the given system of equations is consistent and has infinitely many solutions.

The given system has

(i) no solution when $\lambda=5$ and $\mu\neq 9$

(ii) unique solution when $\lambda\neq 5$ and $\mu\in R$

(iii) infinitely many solution when $\lambda=5$ and $\mu=9$.

EXERCISE 1.7

Concept Corner

The homogeneous system of linear equation $AX=O$

where $A=[a_{ij}]_{n\times n}$ and $X=[x_{ij}]_{n\times 1}$ and $O=[o]_{n\times 1}$

(i) has the trivial solution if $|A|\neq 0$ (ii) has a non trivial solution if $|A|=0$

System of Homogeneous Equation

⇩ $|A|\neq 0$ and $\rho(A)=\rho([A|0])=$ number of unknowns ⇩ The system has trivial solution. $x=y=z$

⇩ $|A|=0$ and $\rho(A)=\rho([A|0])<$ number of unknowns ⇩ The system has non-trivial solution

1. **Solve the following system of homogenous equations.**

(i) $3x+2y+7z=0, 4x-3y-2z=0, 5x+9y+23z=0$ 5M

The matrix form of the system is $AX=O$, where

$$A=\begin{bmatrix}3 & 2 & 7\\4 & -3 & -2\\5 & 9 & 23\end{bmatrix}, X=\begin{bmatrix}x\\y\\z\end{bmatrix}, O=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

The augmented matrix is

$$[A|O]=\left[\begin{array}{ccc|c}3 & 2 & 7 & 0\\4 & -3 & -2 & 0\\5 & 9 & 23 & 0\end{array}\right]$$

$$\rightarrow\left[\begin{array}{ccc|c}3 & 2 & 7 & 0\\0 & -17 & -34 & 0\\0 & 17 & 34 & 0\end{array}\right]\begin{array}{l}R_3\rightarrow 3R_2-4R_1\\R_3\rightarrow 3R_3-5R_1\end{array}$$

$$\rightarrow\left[\begin{array}{ccc|c}3 & 2 & 7 & 0\\0 & -17 & -34 & 0\\0 & 0 & 0 & 0\end{array}\right] R_3\rightarrow R_3+R_1$$

$$\rightarrow\left[\begin{array}{ccc|c}3 & 2 & 7 & 0\\0 & 1 & 2 & 0\\0 & 0 & 0 & 0\end{array}\right] R_2\rightarrow -\frac{1}{17}R_2$$

$\therefore \rho(A)=\rho([A|O])=2$ <number of unknowns.

$\therefore$ The system is consistent and has infinite number of (non-trivial) solution.

The equivalent system of equation

$3x+2y+7z=0$ .... (1)

$y+2z=0$ .... (2)

Let $z=t, t\in R$

$(2)\Rightarrow y+2t=0\Rightarrow y=-2t$

$(1)\Rightarrow 3x+2(-2t)+7(t)=0$

$3x-4t+7t=0$

$3x+3t=0$

$x+t=0$

$x=-t$

$\therefore$ The solution is $(x=-t, y=-2t, z=t)$ where $t\in R$

(ii) $2x+3y-z=0, x-y-2z=0, 3x+y+3z=0$ 2M

The matrix form of the system is $AX=O$, where

$$A=\begin{bmatrix}2 & 3 & -1\\1 & -1 & -2\\3 & 1 & 3\end{bmatrix}, X=\begin{bmatrix}x\\y\\z\end{bmatrix}, O=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

The augmented matrix is

$$[A|O]=\left[\begin{array}{ccc|c}2&3&-1&0\\1&-1&-2&0\\3&1&3&0\end{array}\right]$$

$$\rightarrow\left[\begin{array}{ccc|c}1&-1&-2&0\\2&3&-1&0\\3&1&3&0\end{array}\right] R_1\leftrightarrow R_2$$

$$\rightarrow\left[\begin{array}{ccc|c}1&-1&-2&0\\0&5&3&0\\0&4&9&0\end{array}\right] \begin{array}{l}R_2\rightarrow R_2-2R_1\\R_3\rightarrow R_3-3R_1\end{array}$$

$$\rightarrow\left[\begin{array}{ccc|c}1&-1&-2&0\\0&5&3&0\\0&0&33&0\end{array}\right] R_3\rightarrow 5R_3-4R_2$$

$\rho(A)=\rho([A|O])=3=$number of unknowns.

∴ The given system is consistent and has a trivial solution.

∴ The trivial solution is $(x=0,y=0,z=0)$

2. Determine the values of λ for which the following system of equations $x+y+3z=0$, $4x+3y+\lambda z=0$, $2x+y+2z=0$ has (i) a unique solution (ii) a non-trivial solution.

$x+y+3z=0, 4x+3y+\lambda z=0, 2x+y+2z=0$ PTA-4 5M

The matrix form of the system is $AX=O$, where

$$A=\begin{bmatrix}1&1&3\\4&3&\lambda\\2&1&2\end{bmatrix}, X=\begin{bmatrix}x\\y\\z\end{bmatrix}, O=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

The augmented matrix is

$$[A|O]=\left[\begin{array}{ccc|c}1&1&3&0\\4&3&\lambda&0\\2&1&2&0\end{array}\right]$$

$$\rightarrow\left[\begin{array}{ccc|c}1&1&3&0\\0&-1&\lambda-12&0\\0&-1&-4&0\end{array}\right] \begin{array}{l}R_2\rightarrow R_2-4R_1\\R_3\rightarrow R_3-2R_1\end{array}$$

$$\rightarrow\left[\begin{array}{ccc|c}1&1&3&0\\0&-1&\lambda-12&0\\0&0&8-\lambda&0\end{array}\right] R_3\rightarrow R_3-R_2$$

Case (i)

If $\lambda\neq 8$, then $\rho(A)=\rho([A|O])=3=$ number of unknowns

∴ The given system of equations is consistent and has a unique solution or trivial solution.

Case (ii)

If $\lambda=8$, then $\rho(A)=\rho([A|0])=2<$ number of unknowns

∴ The given system is consistent and has a non-trival solution.

The given system has

(i) a unique solution when $\lambda\neq 8$

(ii) a non-trivial solution when $\lambda=8$

3. **By using Gaussian elimination method, balance the chemical reaction equation:** $\boldsymbol{C_2H_6 + O_2 \rightarrow H_2O + CO_2}$ 5M

We are searching for positive integers x_1, x_2, x_3 and x_4 such that

$$x_1C_2H_6 + x_2O_2 = x_3H_2O + x_4CO_2 \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Equating carbon, Hydrogen and Oxygen atoms on the left-hand side of (1) to the respective carbon, Hydrogen and Oxygen atoms on the right-hand side of (1), we get the system of linear equations.

$$2x_1 = x_4 \quad \Rightarrow 2x_1 - x_4 = 0$$
$$6x_1 = 2x_3 \quad \Rightarrow 3x_1 - x_3 = 0$$
$$2x_2 = x_3 + 2x_4 \Rightarrow 2x_2 - x_3 - 2x_4 = 0$$

The matrix form is $AX = 0$, where $A = \begin{bmatrix} 2 & 0 & 0 & -1 \\ 3 & 0 & -1 & 0 \\ 0 & 2 & -1 & -2 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

The augmented matrix is

$$[A|O] = \begin{bmatrix} 2 & 0 & 0 & -1 \\ 3 & 0 & -1 & 0 \\ 0 & 2 & -1 & -2 \end{bmatrix}$$

$$\rightarrow \left[\begin{array}{cccc|c} 2 & 0 & 0 & -1 & 0 \\ 0 & 0 & -2 & 3 & 0 \\ 0 & 2 & -1 & -2 & 0 \end{array}\right] R_2 \rightarrow 2R_2 - 3R_1$$

$$\rightarrow \left[\begin{array}{cccc|c} 2 & 0 & 0 & -1 & 0 \\ 0 & 2 & -1 & -2 & 0 \\ 0 & 0 & -2 & 3 & 0 \end{array}\right] R_2 \leftrightarrow R_3$$

$\rho(A) = \rho([A|O]) = 3 <$ no. of unknowns

$\therefore$ The system is consistent and has infinite number of solutions. The equivalent system of equations:

$$2x_1 - x_4 = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$
$$2x_2 - x_3 - 2x_4 = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$
$$-2x_3 + 3x_4 = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Let $x_4 = t, t \in R - \{0\}$

$(1) \Rightarrow 2x_1 - t = 0 \Rightarrow x_1 = \dfrac{t}{2}$

$(3) \Rightarrow -2x_3 + 3t = 0 \Rightarrow x_3 = \dfrac{3t}{2}$

$(2) \Rightarrow -2x_2 - \frac{3t}{2} - 2t = 0$

$$-2x_2 - \frac{7t}{2} = 0$$

$$-2x_2 = \frac{7t}{2} \Rightarrow x_2 = \frac{7t}{4}$$

Let us choose $t = 4$

$x_1 = \frac{4}{2} = 2$, $\quad x_2 = \frac{7\times4}{4} = 7$, $\quad x_3 = \frac{3\times4}{2} = 6$, $\quad x_4 = 4$

So, the balanced equation is $2C_2H_6 + 7O_2 \rightarrow 6H_2O + 4CO_2$

EXERCISE 1.8

Choose the correct answer:

1. If $|adj(adjA)| = |A|^9$, then the order of the square matrix A is

(1) 3 **(2) 4** (3) 2 (4) 5

We know that $|adj(adjA)| = |A|^{(n-1)^2}$

where A is a non-singular matrix of order n.

So, $(n-1)^2 = 9$

$\Rightarrow n - 1 = 3$

$\therefore n = 4$

$\therefore$ order of the square matrix A is 4

2. If A is a 3×3 non-singular matrix such that $AA^T = A^TA$ and $B = A^{-1}A^T$, then $BB^T =$

(1) A (2) B **(3) I** (4) B^T

$BB^T = [A^{-1}A^T][A^{-1}A^T]^T$

$= [A^{-1}A^T][(A^T)^T(A^{-1})^T]$ $[\because (AB)^T = B^TA^T]$

$= (A^{-1}A^T)(A(A^T)^{-1})$ $[\because (A^T)^T = A, (A^{-1})^T = (A^T)^{-1}]$

$= A^{-1}(A^TA)(A^T)^{-1}$

$= A^{-1}(AA^T)(A^T)^{-1}$ [Given that $AA^T = A^TA$]

$= (A^{-1}A)[A^T(A^T)^{-1}]$

$= (I)(I)$

$\therefore BB^T = I$

3. If $A = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$, $B = adjA$ and $C = 3A$, then $\frac{|adj\,B|}{|C|} =$

(1) $\frac{1}{3}$ **(2) $\frac{1}{9}$** (3) $\frac{1}{4}$ (4) 1

$\frac{|adj\,B|}{|C|} = \frac{|adj(adjA)|}{|3A|}$

$= \frac{|A|^{(2-1)^2}}{3^2|A|}$ $[\because |adj(adjA)| = |A|^{(n-1)^2}$, here $n = 2]$

$= \frac{|A|}{9|A|} = \frac{1}{9}$ $[|KA| = K^n|A| Here\ n = 2]$

4. If $A\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$, then $A =$

(1) $\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}$ (2) $\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$ **(3) $\begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$** (4) $\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$

$A\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$

$A = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}\frac{1}{6}\begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$

$A = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$

5. If $A = \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$, then $9I - A =$ PTA-2

(1) A^{-1} (2) $\frac{A^{-1}}{2}$ (3) $3A^{-1}$ **(4) $2A^{-1}$**

$A = \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$, $|A| = 14 - 12 = 2$, $adj\ A = \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|} adj\ A = \frac{1}{2} \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$

$2A^{-1} = \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$ .... (1)

$9I - A = 9 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$

$= \begin{bmatrix} 9-7 & 0-3 \\ 0-4 & 9-2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$ .... (2)

From (1) & (2), $9I - A = 2A^{-1}$

6. If $A = \begin{bmatrix} 2 & 0 \\ 1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 4 \\ 2 & 0 \end{bmatrix}$ then $|adj(AB)| =$

(1) -40 **(2) -80** (3) -60 (4) -20

$|A| = 10 - 0 = 10$

$|B| = 0 - 8 = -8$

$|adj(AB| = |(adj\ B)(adjA)|$ $[\because adj(AB) = (adjB)(adjA)]$

$= |adj\ B||adj\ A|$ $[\because |AB| = |A||B|]$

$= |B|^{2-1}|A|^{2-1}$ $[|adj\ A| = |A|^{n-1}]$

$= |B||A|$

$|adj(AB| = (-8)(10) = -80$

7. If $P = \begin{bmatrix} 1 & x & 0 \\ 1 & 3 & 0 \\ 2 & 4 & -2 \end{bmatrix}$ is the adjoint of 3×3 matrix A and $|A| = 4$, then x is

(1) 15 (2) 12 (3) 14 **(4) 11**

$adjA = P = \begin{bmatrix} 1 & x & 0 \\ 1 & 3 & 0 \\ 2 & 4 & -2 \end{bmatrix}$

We know that $|adj\ A| = |A|^{n-1}$ (Here $n = 3$)

$\Rightarrow 1(-6-0) - x(-2-0) + 0 = (4)^{3-1}$

$-6 + 2x = 16$

$2x = 16 + 6 = 22$

$x = \frac{22}{2} = 11$

8. If $A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & -2 & 0 \\ 1 & 2 & -1 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then the value of a_{23} is PTA-5

(1) 0 (2) -2 (3) -3 **(4) -1**

$|A| = 3(2-0) - 1(-2-0) - 1(4+2) = 6 + 2 - 6 = 2$

Co-factor of 2 in A is $= -\begin{vmatrix} 3 & -1 \\ 2 & 0 \end{vmatrix} = -(0+2) = -2$

The value of a_{23} in $A^{-1} = \frac{1}{|A|}$(co-facor of 2 in a_{32} in A)

$= \frac{1}{2}(-2) = -1$

9. If A, B and C are invertible matrices of some order, then which one of the following is not true?

(1) $adjA = |A|A^{-1}$ **(2) $\boldsymbol{adj(AB) = (adjA)(adjB)}$**

(3) $detA^{-1} = (detA)^{-1}$ (4) $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

We know that $adj(AB) = (adjB)(adjA)$

Thus option (2) is not true.

10. If $(AB)^{-1} = \begin{bmatrix} 12 & -17 \\ -19 & 27 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$, then $B^{-1} =$ MAR-20

(1) $\begin{bmatrix} 2 & -5 \\ -3 & 8 \end{bmatrix}$ (2) $\begin{bmatrix} 8 & 5 \\ 3 & 2 \end{bmatrix}$ (3) $\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$ (4) $\begin{bmatrix} 8 & -5 \\ -3 & 2 \end{bmatrix}$

$$(AB)^{-1} = B^{-1}A^{-1} = \begin{bmatrix} 12 & -17 \\ -19 & 27 \end{bmatrix}$$

$$B^{-1}(A^{-1}A) = \begin{bmatrix} 12 & -17 \\ -19 & 27 \end{bmatrix} A$$

$$B^{-1} = \begin{bmatrix} 12 & -17 \\ -19 & 27 \end{bmatrix} (A^{-1})^{-1} \qquad \left[(A^{-1})^{-1} = \frac{1}{|A^{-1}|} adj(A^{-1})\right]$$

$$B^{-1} = \begin{bmatrix} 12 & -17 \\ -19 & 27 \end{bmatrix} \frac{1}{1} \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 36 - 34 & 12 - 17 \\ -57 + 54 & -19 + 27 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -5 \\ -3 & 8 \end{bmatrix}$$

11. If $A^T A^{-1}$ is symmetric, then $A^2 =$

(1) A^{-1} **(2) $(A^T)^2$** (3) A^T (4) $(A^{-1})^2$

Given that $A^T A^{-1}$ is symmetric

$$\therefore (A^T A^{-1})^T = A^T A^{-1}$$

$$(A^{-1})^T (A^T)^T = A^T A^{-1}$$

$$(A^{-1})^T (A) = A^T A^{-1}$$

$$(A^T)^{-1}(A)(A) = A^T (A^{-1}A)$$

$$(A^T)^{-1}A^2 = A^T(I) = A^T$$

$$(A^T)(A^T)^{-1}A^2 = (A^T)(A^T)$$

$$IA^2 = (A^T)^2$$

$$\therefore A^2 = (A^T)^2$$

12. If A is a non-singular matrix such that $A^{-1} = \begin{bmatrix} 5 & 3 \\ -2 & -1 \end{bmatrix}$, then $(A^T)^{-1} =$ PTA-1

(1) $\begin{bmatrix} -5 & 3 \\ 2 & 1 \end{bmatrix}$ (2) $\begin{bmatrix} 5 & 3 \\ -2 & -1 \end{bmatrix}$ (3) $\begin{bmatrix} -1 & -3 \\ 2 & 5 \end{bmatrix}$ **(4) $\begin{bmatrix} 5 & -2 \\ 3 & -1 \end{bmatrix}$**

$$A^{-1} = \begin{bmatrix} 5 & 3 \\ -2 & -1 \end{bmatrix}$$

$$(A^T)^{-1} = (A^{-1})^T = \begin{bmatrix} 5 & -2 \\ 3 & -1 \end{bmatrix}$$

13. If $A = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ x & \frac{3}{5} \end{bmatrix}$ and $A^T = A^{-1}$, then the value of x is

(1) $-\frac{4}{5}$ (2) $-\frac{3}{5}$ (3) $\frac{3}{5}$ (4) $\frac{4}{5}$

$$AA^T = I$$

$$\frac{1}{5}\begin{bmatrix} 3 & 4 \\ 5x & 3 \end{bmatrix}\frac{1}{5}\begin{bmatrix} 3 & 5x \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Multiply and equating a_{12}

$$\frac{1}{25}[15x + 12] = 0$$

$$15x = -12$$

$$x = -\frac{4}{5}$$

PTA-3

14. If $A = \begin{bmatrix} 1 & \tan\frac{\theta}{2} \\ -\tan\frac{\theta}{2} & 1 \end{bmatrix}$ and $AB = I_2$, then $B =$

(1) $\left(\cos^2\frac{\theta}{2}\right)A$ **(2)** $\left(\boldsymbol{\cos^2\frac{\theta}{2}}\right)\boldsymbol{A^T}$ (3) $(\cos^2\theta)I$ (4) $\left(\sin^2\frac{\theta}{2}\right)A$

$$AB = I \Rightarrow B = A^{-1} = \frac{1}{|A|} adj\ A$$

$$B = \frac{1}{1+tan^2\frac{\theta}{2}}\begin{bmatrix} 1 & -tan\frac{\theta}{2} \\ tan\frac{\theta}{2} & 1 \end{bmatrix}$$

$$= \frac{1}{sec^2\frac{\theta}{2}}A^T = \left(cos^2\frac{\theta}{2}\right)A^T$$

15. If $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ and $A(adjA) = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$, then $k =$

(1) 0 (2) $\sin\theta$ (3) $\cos\theta$ **(4) 1**

$$A(adjA) = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} = KI = |A|I$$

$$|A| = K$$

$$\cos^2\theta + sin^2\theta = K$$

$$\therefore k = 1$$

16. If $A = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$ be such that $\lambda A^{-1} = A$, then λ is

(1) 17 (2) 14 **(3) 19** (4) 21

$$\lambda A^{-1} = A$$

$$\lambda(AA^{-1}) = (A)A$$

$$\lambda I = A^2$$

$$\lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$$

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 4+15 & 6-6 \\ 10-10 & 15+4 \end{bmatrix}$$

$$\lambda = 4 + 15 = 19$$

17. If $adj\,A = \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}$ and $adj\,B = \begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix}$ then $adj(AB)$ is

(1) $\begin{bmatrix} -7 & -1 \\ 7 & -9 \end{bmatrix}$ **(2)** $\begin{bmatrix} \mathbf{-6} & \mathbf{5} \\ \mathbf{-2} & \mathbf{-10} \end{bmatrix}$ (3) $\begin{bmatrix} -7 & 7 \\ -1 & -9 \end{bmatrix}$ (4) $\begin{bmatrix} -6 & -2 \\ 5 & -10 \end{bmatrix}$

$adj\,(AB) = (adjB)(adjA)$

$= \begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}$

$= \begin{bmatrix} 2-8 & 3+2 \\ -6+4 & -9-1 \end{bmatrix}$

$= \begin{bmatrix} -6 & 5 \\ -2 & -10 \end{bmatrix}$

18. The rank of the matrix $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ -1 & -2 & -3 & -4 \end{bmatrix}$ is

(1) 1 (2) 2 (3) 4 (4)3

$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ -1 & -2 & -3 & -4 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + R_1 \end{matrix}$

$\therefore \rho(A) = 1$

19. If $x^a y^b = e^m, x^c y^d = e^n, \Delta_1 = \begin{vmatrix} m & b \\ n & d \end{vmatrix}, \Delta_2 = \begin{vmatrix} a & m \\ c & n \end{vmatrix}, \Delta_3 = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$, then the values of x and y are respectively.

PTA-3

(1) $e^{\left(\frac{\Delta_2}{\Delta_1}\right)}, e^{\left(\frac{\Delta_3}{\Delta_1}\right)}$ (2) $\log(\Delta_1/\Delta_3), \log(\Delta_2/\Delta_3)$

(3) $\log(\Delta_2/\Delta_1), \log(\Delta_3/\Delta_1)$ **(4)** $\boldsymbol{e^{\left(\frac{\Delta_1}{\Delta_3}\right)}, e^{\left(\frac{\Delta_2}{\Delta_3}\right)}}$

$x^a y^b = e^m$

$log(x^a y^b) = loge^m$

$a\,logx + b\,logy = m\,loge$

$a\,logx + b\,logy = m$ .... (1)

$x^c y^d = e^n$

$log(x^c y^d) = loge^n$

$c\,logx + d\,logy = n$ .... (2)

By Crammer's rule, we get

$\Delta = \begin{vmatrix} a & b \\ c & d \end{vmatrix}, \quad \Delta_1 = \begin{vmatrix} m & b \\ n & d \end{vmatrix}, \quad \Delta_2 = \begin{vmatrix} a & m \\ c & n \end{vmatrix}$

But, it is given that $\Delta_3 = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

$\therefore \Delta = \Delta_3$

So, $log_e^x = \frac{\Delta_1}{\Delta} = \frac{\Delta_1}{\Delta_3}$

$\Rightarrow x = e^{\left(\frac{\Delta_1}{\Delta_3}\right)}$

$log_e^y = \frac{\Delta_2}{\Delta} = \frac{\Delta_2}{\Delta_3}$

$\Rightarrow y = e^{\left(\frac{\Delta_2}{\Delta_3}\right)}$

20. Which of the following is / are correct?
(i) Adjoint of a symmetric matrix is also a symmetric matrix.
(ii) Adjoint of a diagonal matrix is also a diagonal matrix.
(iii) If A is a square matrix of order n and λ is a scalar, then $adj(\lambda A) = \lambda^n adj(A)$.
(iv) $A(adjA) = (adjA)A = |A|I$
(1) Only (i) (2) (ii) and (iii) (3) (iii) and (iv) **(4) (i), (ii)and (iv)**

$adj\ (\lambda A) = \lambda^{n-1}\ adj\ (A)$ But given that $adj(\lambda A) = \lambda^n adj\ (A)$
So, (iii) only a wrong statement.

21. If $\rho(A) = \rho([A|B])$, then the system $AX = B$ of linear equations is MAR-20 PTA-6
(1) consistent and has a unique solution **(2) consistent**
(3) consistent and has infinitely many solution (4) inconsistent

22. If $0 \le \theta \le \pi$ and the system of equations $x + (\sin\theta)y - (\cos\theta)z = 0$, $(\cos\theta)x - y + z = 0$, $(\sin\theta)\, x + y - z = 0$ has a non-trivial solution then θ is
(1) $\frac{2\pi}{3}$ (2) $\frac{3\pi}{4}$ (3) $\frac{5\pi}{6}$ **(4) $\frac{\pi}{4}$**

The system has a non-trivial solution. $\therefore \Delta = \begin{vmatrix} 1 & sin\ \theta & -cos\ \theta \\ cos\ \theta & -1 & 1 \\ sin\ \theta & 1 & -1 \end{vmatrix} = 0$

$\Rightarrow 1(1-1) - sin\theta(-cos\ \theta - sin\theta) - cos\theta(cos\ \theta + sin\ \theta) = 0$

$\Rightarrow sin\theta cos\theta + sin^2\theta - cos^2\theta - sin\ \theta\ cos\ \theta = 0$

$sin^2\theta - cos^2\theta = 0$

$\frac{sin^2\theta}{cos^2\theta} - \frac{cos^2\theta}{cos^2\theta} = 0$

$\frac{sin^2\theta}{cos^2\theta} - 1 = 0 \Rightarrow \quad tan^2\theta = 1 \Rightarrow \quad \tan\theta = 1$

$\Rightarrow \theta = \frac{\pi}{4} [\because 0 \le \theta \le \pi]$

23. The augmented matrix of a system of linear equations is $\begin{bmatrix} 1 & 2 & 7 & 3 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & \lambda - 7 & \mu + 5 \end{bmatrix}$. The system has infinitely many solutions if
(1) $\lambda = 7, \mu \ne -5$ (2) $\lambda = -7, \mu = 5$ (3) $\lambda \ne 7, \mu \ne -5$ **(4)$\boldsymbol{\lambda = 7, \mu = -5}$**

Let $A = \begin{bmatrix} 1 & 2 & 7 & 3 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & \lambda - 7 & \mu + 5 \end{bmatrix}$

If $\lambda = 7, \mu = -5$ then $\rho(A) = 2, \rho[(A|B)] = 2$

$\therefore \rho(A) = \rho([A|B]) = 2 <$ no. of unknowns

Thus, the given system is consistent and has infinitely many solutions if $\lambda = 7, \mu = -5$

24. Let $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ and $4B = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & x \\ -1 & 1 & 3 \end{bmatrix}$. If B is the inverse of A, then the value of x is
(1) 2 (2) 4 (3) 3 **(4)1**

$AB = \frac{1}{4}I$

Evaluate $a_{23} = 0$ only

$1 + 2x - 3 = 0$

$2x = 2$

$x = 1$

25. If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, then $adj(adjA)$ is PTA-2

(1) $\begin{bmatrix} \mathbf{3} & \mathbf{-3} & \mathbf{4} \\ \mathbf{2} & \mathbf{-3} & \mathbf{4} \\ \mathbf{0} & \mathbf{-1} & \mathbf{1} \end{bmatrix}$ (2) $\begin{bmatrix} 6 & -6 & 8 \\ 4 & -6 & 8 \\ 0 & -2 & 2 \end{bmatrix}$ (3) $\begin{bmatrix} -3 & 3 & -4 \\ -2 & 3 & -4 \\ 0 & 1 & -1 \end{bmatrix}$ (4) $\begin{bmatrix} 3 & -3 & 4 \\ 0 & -1 & 1 \\ 2 & -3 & 4 \end{bmatrix}$

$A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$

$|A| = 3[-3+4] + 3[2-0] + 4[-2-0] = 3(1) + 3(2) + 4(-2) = 3 + 6 - 8 = 1$

$adj(adjA) = |A|^{n-2}A = (1)^{3-2}\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$

$\therefore adj(adjA) = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$

Creative Questions

Choose the correct or the most suitable answer from the given four alternatives

1. If A and B are orthogonal, then $(AB)^T\ (AB)$ is PTA-1

(1) A (2) B **(3) I** (4) A^T

2. The adjoint of 3×3 matrix P is $\begin{bmatrix} -1 & 2 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, then the possible values of the determinant P is (are) PTA-4

(1) 3 (2) -3 **(3) ± 3** (4) $\pm\sqrt{3}$

$adj\ P = \begin{bmatrix} -1 & 2 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$

$|adj\ P| = -1(1-4) - 2(1-4) + 2(2-2)$

$= -1(-3) - 2(-3) + 0 = 3 + 6 = 9$

$|P| = \pm\sqrt{|adj\ P|} = \pm\sqrt{9}$

$|P| = \pm 3$

3. If A is a 3×3 matrix such that $|3adjA| = 3$ then $|A|$ is equal to PTA-5

(1) $\frac{1}{3}$ (2) $-\frac{1}{3}$ **(3) $\pm\frac{1}{3}$** (4) ± 3

$|3adjA| = 3$ $(\because |kA| = k^3|A|)$

$3^3|adjA| = 3$

$27\ |A|^{3-1} = 3$ $(\because |adj\ A| = |A|^{n-1})$

$|A|^2 = \frac{3}{27} = \frac{1}{9} \Rightarrow |A| = \pm\frac{1}{3}$

4. Let A be a non-singular matrix then which one of the following is false PTA-6

(1) $(adjA)^{-1} = \frac{A}{|A|}$ (2) I is an orthogonal matrix

(3) $adj(adjA) = |A|^n A$ (4) If A is symmetric then $adjA$ is symmetric

5. If for a matrix A, $|A| = 6$ and $adj\ A = \begin{bmatrix} 1 & -2 & 4 \\ 4 & 1 & 1 \\ -1 & \lambda & 0 \end{bmatrix}$, then λ is equal to

(1) -1 (2) 0 (3) 1 **(4) 2**

We know that $|adj\ A| = |A|^{3-1} = |A|^2$

$1(0-\lambda) + 2(0+1) + 4(4\lambda + 1) = 6^2$

$-\lambda + 2 + 16\lambda + 4 = 36$

$15\lambda + 6 = 36$

$\Rightarrow 15\lambda = 36 - 6 = 30$

$\Rightarrow \quad \lambda = \frac{30}{15} = 2$

6. If $adj\ A = \begin{pmatrix} -2 & 1 \\ 4 & 3 \end{pmatrix}$ and $adj\ B = \begin{pmatrix} 4 & -5 \\ 1 & 7 \end{pmatrix}$ then AB is

(1) $\begin{pmatrix} \mathbf{22} & \mathbf{11} \\ \mathbf{-26} & \mathbf{-28} \end{pmatrix}$ (2) $\begin{pmatrix} -28 & 11 \\ -26 & 22 \end{pmatrix}$ (3) $\begin{pmatrix} 22 & -26 \\ 11 & -28 \end{pmatrix}$ (4) $\begin{pmatrix} -22 & -11 \\ 26 & 28 \end{pmatrix}$

$adj\ (AB) = (adjB)\ (adjA)$

$= \begin{pmatrix} 4 & -5 \\ 1 & 7 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} -8-20 & 4-15 \\ -2+28 & 1+21 \end{pmatrix}$

$= \begin{pmatrix} -28 & -11 \\ 26 & 22 \end{pmatrix}$

$\therefore AB = \begin{pmatrix} 22 & 11 \\ -26 & -28 \end{pmatrix}$

7. If $A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ then $A\ (adj\ A) = ?$

(1) $\begin{bmatrix} -\cos\theta & \sin\theta & 0 \\ -\sin\theta & -\cos\theta & 0 \\ 0 & 0 & -1 \end{bmatrix}$ **(2)** $\begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}$ (3) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ (4) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

$|A| = \cos\theta\,[\cos\theta - 0] + \sin\theta\ [\sin\theta - 0]$

$= \cos^2\theta + \sin^2\theta = 1$

$|A| = 1$

$A\ (adj\ A) = |A| I_3 = 1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

8. If the matrix $\begin{bmatrix} x+2 & 0 \\ x^4 & x-1 \end{bmatrix}$ has no inverse then the value of $x =$

(1) 1 (2) 2 **(3) 1** (4) 1 and 2

Let $A = \begin{bmatrix} x+2 & 0 \\ x^4 & x-1 \end{bmatrix}$

If A^{-1} does not exist, then $|A| = 0$

$\Rightarrow (x+2)\,(x-1) - 0 = 0$

$(x+2)\,(x-1) = 0$

$\therefore x = -2\ \text{(or)}\ x = 1$

9. If A and B are two non-singular matrices then $|B^{-1}AB| =$

(1) A **(2) $|A|$** (3) $|B|$ (4) $|B^{-1}|$

$$|B^{-1}AB| = |B^{-1}|\,|A|\,|B|$$
$$= |B^{1}B|\,|A|$$
$$= |I||A|$$
$$= (1)|A|$$
$$= |A|$$

10. If A is a non – singular matrix, then $|A^{-1}| =$

(1) $\left|\frac{1}{A^2}\right|$ (2) $\frac{1}{|A^2|}$ (3) $\left|\frac{1}{A}\right|$ **(4) $\frac{1}{|A|}$**

11. If $A = \begin{bmatrix} 1 & 1 & 1 \\ 5 & 5 & 5 \\ 8 & 8 & 8 \end{bmatrix}$ then

(1) $A^2 = 9A$ (2) $A^2 = 27A$ (3) $A + A = A^2$ **(4) A^{-1}does not exist**

$|A| = 1(40 - 40) - 1(40 - 40) + 1(40 - 40)$

$|A| = 0$

$\therefore A^{-1}$does not exist

12. If $A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$ then $|(A^{-1})^{-1}|$

(1) 11 (2) -11 (3) $\frac{1}{11}$ (4) None

$|A| = 8 + 3 = 11$

$|(A^{-1})^{-1}| = |A| = 11$

13. If $A = \begin{bmatrix} -1 & 3 \\ 2 & 1 \end{bmatrix}$ then $I + A + A^2 + \cdots \infty =$

(1) $\begin{bmatrix} 0 & 3 \\ 2 & 3 \end{bmatrix}$ **(2) $\frac{1}{6}\begin{bmatrix} 0 & -3 \\ -2 & -2 \end{bmatrix}$** (3) $\begin{bmatrix} 0 & -3 \\ -2 & 2 \end{bmatrix}$ (4) $\frac{1}{6}\begin{bmatrix} 0 & 3 \\ -2 & 2 \end{bmatrix}$

$$I - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -2 & 0 \end{bmatrix}$$

$|I - A| = 0 - 6 = -6$

$(I - A)^{-1} = \frac{1}{|I-A|}\, adj\,(I - A)$

$\therefore 1 + A + A^2 + \cdots + \infty = \frac{1}{-6}\begin{bmatrix} 0 & 3 \\ 2 & 2 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 0 & -3 \\ -2 & -2 \end{bmatrix}$

14. If $A = \begin{bmatrix} 5 & -2 \\ 3 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 \\ -3 & 5 \end{bmatrix}$ then A and B are

(1) Non- singular matrices (2) Square matrices

(3) Inverse to each other **(4) All of these**

$|A| = -5 + 6 = 1, |B| = -5 + 6 = 1$

$\Rightarrow$ A and B are non – singular matrices. A and B are square matrices

$A^{-1} = \frac{1}{|A|}\, adj\, A = \frac{1}{1}\begin{bmatrix} -1 & 2 \\ -3 & 5 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ -3 & 5 \end{bmatrix} = B$

$\therefore$ A and B both are inverse each other.

Thus , Answer is " all of these"

15. If $10A - 50I = 0$ and $A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ then A^{-1} is

(1) $5I$ (2) $-\frac{1}{5}I$ **(3) $\frac{1}{5}I$** (4) $-5I$

$10A - 50I = 0$

Pre – multiplying by A^{-1}, we get

$10(A^{-1}A) - 50(A^{-1}I) = 0$

$10I - 50A^{-1} = 0$

$10I = 50A^{-1}$

$A^{-1} = \frac{10}{50}I = \frac{1}{5}I$

16. $adj\ (I_2) = ?$

(1) $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ **(2) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$** (3) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ (4) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$adj\ (I_n) = I_n$

$\Rightarrow adj\ (I_2) = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

17. If $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ and $|adj\ A| = 81$ then $|A|$ is

(1) 9 (2) 81 (3) 18 **(4) ± 9**

$|adj\ A| = |A^2| = 81$

$\Rightarrow |A| = \pm 9$

18. If $A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$ *then* $|adj\ (4A)|$ is

(1) 44 (2) 11 **(3) 176** (4) 121

$|A| = 8 + 3 = 11$

$|adj\ (4A)| = k^{n(n-1)}|A|^{n-1}$

$|adj\ (4A)| = 4^{2(2-1)}|A|^{2-1}$

$= 4^2|A|$

$= 16[11] = 176$

(OR) $adj\ \lambda A = \lambda^{n-1} adjA$

$|adj\ KA| = |K^1 adj\ A|$

$= 4^2|adj\ A| = 16|A|$

$= 16 \times 11$

$= 176$

19. If A is 3×3 matrix such that $|5\ adj\ A| = 5$ then $|A|$ is equal to

(1) $\pm\frac{1}{5}$ (2) ± 5 (3) ± 1 (4) $\pm\frac{1}{25}$

$|5\ adj\ A| = 5$

$\Rightarrow 5^3|adj\ A| = 5$

$\Rightarrow 125\ |A|^2 = 5$

$\Rightarrow |A|^2 = \frac{5}{125} = \frac{1}{25}$

$|A| = \pm\frac{1}{5}$

20. The rank of the matrix $\begin{bmatrix} 2 & 5 \\ -3 & 4 \end{bmatrix}$ is

(1) 1 **(2) 2** (3) 3 (4) 5

Let $A = \begin{bmatrix} 2 & 5 \\ -3 & 4 \end{bmatrix}$

$|A| = 8 + 15 = 23 \neq 0$

$\therefore \rho(A) = 2$

21. If $k = 5$ then the rank of the matrix $A = \begin{bmatrix} 5 & k & k \\ k & 5 & k \\ k & k & 5 \end{bmatrix}$ is

(1) 1 (2) 2 (3) 3 (4) 0

$A = \begin{bmatrix} 5 & k & k \\ k & 5 & k \\ k & k & 5 \end{bmatrix} \sim \begin{bmatrix} 5 & 5 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_2$

$\rho(A) = 1$

22. The system of equations $x - 2y + z = 0, y - z = 3, 2x - 3z = 10$ is

(1) Inconsistent (2) consistent

(3) consistent and has infinite number of solutions

(4) consistent and has unique solution.

Matrix form of the equation is

$$\underset{A}{\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 2 & 0 & -3 \end{bmatrix}} \underset{X}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \underset{B}{\begin{bmatrix} 0 \\ 3 \\ 10 \end{bmatrix}}$$

$|A| = 1(-3 - 0) + 2(0 + 2) + 1(0 - 2)$

$= -3 + 4 - 2 = -1 \neq 0$

$\therefore$ The system is consistent and has unique solution.

23. If the rank of the matix $\begin{bmatrix} \lambda & -2 & 0 \\ 0 & \lambda & -2 \\ -2 & 0 & \lambda \end{bmatrix}$ is 2, then λ is

(1) 1 **(2) 2** (3) 3 (4) 4

Let $A = \begin{bmatrix} \lambda & -2 & 0 \\ 0 & \lambda & -2 \\ -2 & 0 & \lambda \end{bmatrix}$

If $\rho(A) = 2$, then $|A| = 0$

$\Rightarrow \lambda(\lambda^2 - 0) + 2(0 - 4) + 0 = 0$

$\Rightarrow \lambda^3 - 8 = 0$

$\Rightarrow \lambda^3 = 8 = 2^3$

$\Rightarrow \lambda = 2$

24. If $\rho(A) = \rho([(A|B)]) = 3$ the numbers of unknowns then the system is

(1) Consistent

(2) inconsistent

(3) consistent and has unique solution

(4) consistent and has infinitely many solutions

2 Marks

1. If $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}, B = \begin{bmatrix} 4 & 0 \\ 2 & 5 \end{bmatrix}$, find $adj\ (AB)$. PTA-3

$$AB = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 2 & 5 \end{bmatrix}$$
$$= \begin{bmatrix} 14 & 15 \\ 8 & 10 \end{bmatrix}$$
$$adj(AB) = \begin{bmatrix} 10 & -15 \\ -8 & 14 \end{bmatrix}$$

2. If A is a non-singular matrix of odd order, prove that $|adj(A)|$ is positive. PTA-4

Let A be a non-singular matrix of order 2m+1 where m=0, 1, 2……….

$\therefore\ |A| \neq 0$

We know $|Adj\ A| = |A|^{n-1}\ (n = 2m+1)$

$= |A|^{2m} > 0$

$\therefore |adj\ A|$ is positive.

3. If $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$ then find $|adj\ A|$

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$
$$|A| = 1(1-0) - 0 + 2(4-3)$$
$$= 1 - 0 + 2 = 3$$
$$|adj\ A| = |A|^{3-1} \qquad [\because |adj\ A| = |A|^{n-1}]$$
$$= |A|^2 = 3^2 = 9$$

4. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 2 & 4 & 3 \end{bmatrix}$ find $|adj\ (3A)|$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 2 & 4 & 3 \end{bmatrix}$$
$$|A| = 1(15-0) - 2(0-0) + 3(0-10)$$
$$= 15 - 0 - 30 = -15$$
$$|adj\ (3A)| = 3^{3(3-1)}(-15)^{3-1}$$
$$[\because |adj\ (kA)| = k^{n(n-1)}|A|^{n-1}]$$
$$= 3^6(-15)^2$$
$$= 729 \times 225 = 164025$$

5. If $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 3 & 0 \\ 2 & 2 & 4 \end{bmatrix}$ find $|adj\ (adj\ A)|$

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 3 & 0 \\ 2 & 2 & 4 \end{bmatrix}$$
$$|A| = 1(12-0) + 1(4-0) + 1(2-6)$$
$$= 12 + 4 - 4 = 12$$
$$|adj\ (adj\ A)| = |A|^{(n-1)^2}$$
$$= (12)^{(3-1)^2}$$
$$= 12^4$$
$$= 20736$$

6. If $B = \begin{bmatrix} 4 & 3 \\ 5 & -1 \end{bmatrix}$ then find $|B\ (adjB)|$

$$B = \begin{bmatrix} 4 & 3 \\ 5 & -1 \end{bmatrix}$$
$$[\because |AB| = |A||B| \quad , \quad |adj\ A| = |A|^{n-1}]$$
$$|B| = -4 - 15 = -19$$
$$|B\ (adjB)| = |B||adj\ B|$$
$$= |B||B|^{2-1}$$
$$= |B|^2$$
$$= (-19)^2$$
$$= 361$$

7. If $B = \begin{bmatrix} 1 & x & 3 \\ 1 & 3 & 3 \\ 2 & 4 & 4 \end{bmatrix}$ is the adjoint of a 3×3 matrix A and $|A| = 4$, then find x

Given: $B = adj\ A$

$$|B| = |adj\ A| = |A|^2$$
$$1(12-12) - x(4-6) + 3(4-6) = 4^2$$
$$0 + 2x - 6 = 16$$
$$2x = 16 + 6$$
$$2x = 22$$
$$x = \frac{22}{2}$$
$$x = 11$$

8. If $A=\begin{bmatrix}1&2&3\\3&1&2\\1&0&3\end{bmatrix}$ find $(adj\ A)A$

$$A=\begin{bmatrix}1&2&3\\3&1&2\\1&0&3\end{bmatrix}$$

$$|A|=1(3-0)-2(9-2)+3(0-1)$$
$$=3-14-3$$
$$=-14$$

$$(adj\ A)A=|A|I$$
$$=-14\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$
$$=\begin{bmatrix}-14&0&0\\0&-14&0\\0&0&-14\end{bmatrix}$$

9. Find the rank of the following matrices by minor method:

(i) $\begin{bmatrix}1&2&3\\2&3&4\end{bmatrix}$ (ii) $\begin{bmatrix}1&3\\2&-1\\-1&-3\end{bmatrix}$

(i) Let $A=\begin{bmatrix}1&2&3\\2&3&4\end{bmatrix}$

Order of the matrix A is 2×3

$\therefore \rho(A)\le \min\{2,3\}=2$

We find that the second order minor

$$\begin{vmatrix}1&2\\2&3\end{vmatrix}=3-4=-1\ne 0$$

$\rho(A)=2$

(ii) Let $A=\begin{bmatrix}1&3\\2&-1\\-1&-3\end{bmatrix}$

Order of the matrix A is 3×2

$\therefore \rho(A)\le \min\{3,2\}=2$

We find that the second order minor

$$\begin{vmatrix}1&3\\2&-1\end{vmatrix}=-1-6$$
$$=-7\ne 0$$

$\rho(A)=2$

10. Solve the following system of linear equations by using Cramer's Rule

$$\mathbf{3x-y=3\,,2x+y=7}$$

$$\Delta=\begin{vmatrix}3&-1\\2&1\end{vmatrix}=3+2=5$$

$\Delta=5$

$$\Delta_1=\begin{vmatrix}3&-1\\7&1\end{vmatrix}=3+7=10$$

$\Delta_1=10$

$$\Delta_2=\begin{vmatrix}3&3\\2&7\end{vmatrix}=21-6=15$$

$\Delta_2=15$

By Cramer's Rule,

$$x=\frac{\Delta_1}{\Delta}=\frac{10}{5}=2$$

$$y=\frac{\Delta_2}{\Delta}=\frac{15}{5}=3$$

The solution is $x=2, y=3$

11. Solve the following system of linear equation by matrix inversion method.

$$\mathbf{2x+3y=23,\ 3x+4y=32}$$

The matrix form is $AX=B$, where

$$A=\begin{bmatrix}2&3\\3&4\end{bmatrix}, X=\begin{bmatrix}x\\y\end{bmatrix}, B=\begin{bmatrix}23\\32\end{bmatrix}$$

$|A|=8-9=-1\ne 0$

A^{-1} exists

$$adj\ A=\begin{bmatrix}4&-3\\-3&2\end{bmatrix}$$

$$A^{-1}=\frac{1}{|A|}adj\ A$$

$$A^{-1}=\frac{1}{-1}\begin{bmatrix}4&-3\\-3&2\end{bmatrix}$$
$$=\begin{bmatrix}-4&3\\3&-2\end{bmatrix}$$

$AX=B$

$$X=A^{-1}B=\begin{bmatrix}-4&3\\3&-2\end{bmatrix}\begin{bmatrix}23\\32\end{bmatrix}$$

$$\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}-92+96\\69-64\end{bmatrix}$$

$$\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}4\\5\end{bmatrix}$$

The solution is $(x,y)=(4,5)$

3 Marks

1. If A= $\begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$, find the value of λ so that $A^2 = \lambda A - 2I$. PTA-2

$$A = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 9-8 & -6+4 \\ 12-8 & -8+4 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 4 & -4 \end{bmatrix}$$

$$\lambda A - 2I = \begin{bmatrix} 3\lambda & -2\lambda \\ 4\lambda & -2\lambda \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3\lambda - 2 & -2\lambda \\ 4\lambda & -2\lambda - 2 \end{bmatrix}$$

Given, $A^2 = \lambda A - 2I$

$$\therefore \begin{bmatrix} 1 & -2 \\ 4 & -4 \end{bmatrix} = \begin{bmatrix} 3\lambda - 2 & -2\lambda \\ 4\lambda & -2\lambda - 2 \end{bmatrix}$$

Equating the corresponding elements

$$\therefore 4\lambda = 4$$

$$\lambda = 1$$

2. Find the rank of the matrix $\begin{bmatrix} 4 & -2 & 6 & 8 \\ 1 & 1 & -3 & -1 \\ 15 & -3 & 9 & 21 \end{bmatrix}$ PTA-4

$$A = \begin{bmatrix} 4 & -2 & 6 & 8 \\ 1 & 1 & -3 & -1 \\ 15 & -3 & 9 & 21 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -3 & -1 \\ 4 & -2 & 6 & 8 \\ 15 & -3 & 9 & 21 \end{bmatrix} R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & 1 & -3 & -1 \\ 0 & -6 & 18 & 12 \\ 0 & -18 & 54 & 36 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 4R_1 \\ R_3 \to R_3 - 15R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -3 & -1 \\ 0 & -6 & 18 & 12 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_3 \to R_3 - 3R_2$$

$$\rho(A) = 2$$

3. Solve by matrix inversion method: $5x + 2y = 4, 7x + 3y = 5$. PTA-5

$$\begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$AX = B$$

$$X = A^{-1}B$$

$|A| = 1$

$$adj\ A = \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\ adj\ A$$

$$= \frac{1}{1}\begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}$$

$$X = A^{-1}B$$

$$= \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$x = 2$ & $y = -3$

4. Find the adjoint of the matrix $A = \begin{bmatrix} 1 & 3 \\ 2 & -5 \end{bmatrix}$ and verify that $A(adjA) = (adjA)A = |A|I$ PTA-6

$$A = \begin{bmatrix} 1 & 3 \\ 2 & -5 \end{bmatrix} |A| = -11$$

$$adj\ A = \begin{bmatrix} -5 & -3 \\ -2 & 1 \end{bmatrix}$$

$$A\ (adj\ A) = \begin{bmatrix} 1 & 3 \\ 2 & -5 \end{bmatrix}\begin{bmatrix} -5 & -3 \\ -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -11 & 0 \\ 0 & -11 \end{bmatrix}$$

$$(adj\ A)\ A = \begin{bmatrix} -5 & -3 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} -11 & 0 \\ 0 & -11 \end{bmatrix}$$

$$|A|I = -11\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]$$

$$= \begin{bmatrix} -11 & 0 \\ 0 & -11 \end{bmatrix}$$

$$\therefore A\ (adj\ A) = (adj\ A)A = |A|I$$

5. Find the adjoint of $\frac{1}{5}\begin{bmatrix}2&5&3\\3&1&2\\1&2&1\end{bmatrix}$

Let $B=\frac{1}{5}\begin{bmatrix}2&5&3\\3&1&2\\1&2&1\end{bmatrix}$

Let us consider $A=\begin{bmatrix}2&5&3\\3&1&2\\1&2&1\end{bmatrix}$

$$adj\ A=\begin{bmatrix}\begin{vmatrix}1&2\\2&1\end{vmatrix}&-\begin{vmatrix}3&2\\1&1\end{vmatrix}&\begin{vmatrix}3&1\\1&2\end{vmatrix}\\-\begin{vmatrix}5&3\\2&1\end{vmatrix}&\begin{vmatrix}2&3\\1&1\end{vmatrix}&-\begin{vmatrix}2&5\\1&2\end{vmatrix}\\\begin{vmatrix}5&3\\1&2\end{vmatrix}&-\begin{vmatrix}2&3\\3&2\end{vmatrix}&\begin{vmatrix}2&5\\3&1\end{vmatrix}\end{bmatrix}^T$$

$$=\begin{bmatrix}(1-4)&-(3-2)&(6-1)\\-(5-6)&(2-3)&-(4-5)\\(10-3)&-(4-9)&(2-15)\end{bmatrix}^T$$

$$=\begin{bmatrix}-3&-1&5\\1&-1&1\\7&5&-13\end{bmatrix}^T$$

$$adj\ A=\begin{bmatrix}-3&1&7\\-1&-1&5\\5&1&-13\end{bmatrix}$$

$$adj\ B=adj\ \left(\frac{1}{5}A\right)$$

$$=\left(\frac{1}{5}\right)^{3-1}adj\ A$$

$[\because adj\ (kA)=k^{n-1}adj\ A]$

$$adj\ B=\frac{1}{25}\begin{bmatrix}-3&1&7\\-1&-1&5\\5&1&-13\end{bmatrix}$$

6. If $A=\begin{bmatrix}2&-3\\1&5\end{bmatrix}$ prove that $adj(A^T)=(adj\ A)^T$

$A=\begin{bmatrix}2&-3\\1&5\end{bmatrix}$, $\quad A^T=\begin{bmatrix}2&1\\-3&5\end{bmatrix}$

$adj\ (A^T)=\begin{bmatrix}5&-1\\3&2\end{bmatrix}$ .... (1)

$A=\begin{bmatrix}2&-3\\1&5\end{bmatrix}$

$adj\ A=\begin{bmatrix}5&3\\-1&2\end{bmatrix}$

$(adj\ A)^T=\begin{bmatrix}5&-1\\3&2\end{bmatrix}$ .... (2)

From (1) and (2), $\quad adj(A^T)=(adj\ A)^T$

7. If $A=\begin{bmatrix}2&-3\\4&6\end{bmatrix}$, verify that $(adj\ A)^{-1}=adj\ (A^{-1})$

$A=\begin{bmatrix}2&-3\\4&6\end{bmatrix}$,

$|A|=12+12=24\neq 0$

$\therefore A^{-1}$ exists

$adj\ A=\begin{bmatrix}6&3\\-4&2\end{bmatrix}$

$|adj\ A|=12+12=24$

$adj(adj\ A)=\begin{bmatrix}2&-3\\4&6\end{bmatrix}$

$(adj\ A)^{-1}=\frac{1}{|adj\ A|}\ adj\ (adj\ A)$

$(adj\ A)^{-1}=\frac{1}{24}\begin{bmatrix}2&-3\\4&6\end{bmatrix}$ .... (1)

$A^{-1}=\frac{1}{|A|}adj\ A$

$A^{-1}=\frac{1}{24}\begin{bmatrix}6&3\\-4&2\end{bmatrix}$

$adj\ (A^{-1})=\frac{1}{24}\begin{bmatrix}2&-3\\4&6\end{bmatrix}$ .... (2)

From (1) and (2), we get

$$(adj\ A)^{-1}=adj(A^{-1})$$

8. Solve the following system of linear equations by matrix inversion method $ax+by=a$, $ay-bx=b$

The matrix form is $AX=B$,

where $A=\begin{pmatrix}a&b\\-b&a\end{pmatrix}$, $X=\begin{pmatrix}x\\y\end{pmatrix}$, $B=\begin{pmatrix}a\\b\end{pmatrix}$

$|A|=a^2+b^2\neq 0 \qquad \therefore A^{-1}$ exists

$adj\ A=\begin{pmatrix}a&-b\\b&a\end{pmatrix}$

$A^{-1}=\frac{1}{|A|}\ adj\ A=\frac{1}{a^2+b^2}\begin{pmatrix}a&-b\\b&a\end{pmatrix}$

$AX=B\Rightarrow X=A^{-1}B$

$$=\frac{1}{a^2+b^2}\begin{pmatrix}a&-b\\b&a\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}$$

$$=\frac{1}{a^2+b^2}\begin{pmatrix}a^2-b^2\\ab+ab\end{pmatrix}$$

$$\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}\frac{a^2-b^2}{a^2+b^2}\\\frac{2ab}{a^2+b^2}\end{pmatrix}$$

The solution is $x=\frac{a^2-b^2}{a^2+b^2}$, $y=\frac{2ab}{a^2+b^2}$

9. Send the message "BLASTED" by using the encoding matrix $\begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{-1} & \mathbf{2} & \mathbf{1} \\ \mathbf{0} & \mathbf{-1} & \mathbf{0} \end{bmatrix}$

As the encoded message

BLASTED =(BLA)(STE)(D)

Uncoded row matrices = $(2 \quad 12 \quad 1)(19 \quad 20 \quad 5)(4 \quad 0 \quad 0)$

Uncoded row matrix	Encoding matrix	Coded row matrix
$[2 \quad 12 \quad 1]$	$\begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \\ 0 & -1 & 0 \end{bmatrix}$	$= [2-12 \quad 24-1 \quad 2+12]$ $= [-10 \quad 23 \quad 14]$
$[19 \quad 20 \quad 5]$	$\begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \\ 0 & -1 & 0 \end{bmatrix}$	$= [19-20 \quad 40-5 \quad 19+20]$ $= [-1 \quad 35 \quad 39]$
$[4 \quad 0 \quad 0]$	$\begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \\ 0 & -1 & 0 \end{bmatrix}$	$= [4+0+0 \quad 0+0+0 \quad 4+0+0]$ $= [4 \quad 0 \quad 4]$

$\therefore$ The encoded message is $[-10 \quad 23 \quad 14][-1 \quad 35 \quad 39][4 \quad 0 \quad 4]$

10. Find the rank of the following matrix by minor method. $\begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{3} \\ \mathbf{2} & \mathbf{-1} & \mathbf{3} & \mathbf{4} \\ \mathbf{5} & \mathbf{-1} & \mathbf{7} & \mathbf{11} \end{bmatrix}$

Let $A = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & -1 & 3 & 4 \\ 5 & -1 & 7 & 11 \end{bmatrix}$, Order of the matrix A is $3 \times 4 \Rightarrow \rho(A) \leq \min\{3,4\} = 3$

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 5 & -1 & 7 \end{vmatrix} = 1(-7+3) - 1(14-15) + 1(-2+5)$$
$$= -4 + 1 + 3 = 0$$

$$\begin{vmatrix} 1 & 1 & 3 \\ 2 & -1 & 4 \\ 5 & -1 & 11 \end{vmatrix} = 1(-11+4) - 1(22-20) + 3(-2+5)$$
$$= -7 - 2 + 9 = 0$$

$$\begin{vmatrix} 1 & 1 & 3 \\ 2 & 3 & 4 \\ 5 & 7 & 11 \end{vmatrix} = 1(33-28) - 1(22-20) + 3(14-15)$$
$$= 5 - 2 - 3 = 0$$

$$\begin{vmatrix} 1 & 1 & 3 \\ -1 & 3 & 4 \\ -1 & 7 & 11 \end{vmatrix} = 1(33-28) - 1(-11+4) + 3(-7+3)$$
$$= 5 + 7 - 12 = 0$$

All the minor of order 3 vanishes, $\rho(A) \neq 3$

We find that the second order minor $\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = -1 - 2 = -3 \neq 0 \qquad \rho(A) = 2$

11. Find the inverse using by Gauss – Jordan Method: $\begin{bmatrix} \mathbf{1} & \mathbf{3} & \mathbf{0} \\ \mathbf{-2} & \mathbf{3} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} \end{bmatrix}$

Let $A = \begin{bmatrix} 1 & 3 & 0 \\ -2 & 3 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

Applying Gauss Jordan method, we get $[A \mid I_3] = \left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ -2 & 3 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 9 & 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) R_2 \rightarrow R_2 + 2R_1$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 9 & 1 & 2 & 1 & 0 \end{array}\right) R_2 \leftrightarrow R_3$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & -3 & 1 & 0 & -3 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & -8 & 2 & 1 & -9 \end{array}\right) \begin{array}{l} R_1 \rightarrow R_1 - 3R_2 \\ \\ R_3 \rightarrow R_3 - 9R_2 \end{array}$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & -3 & 1 & 0 & -3 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{8} & \frac{9}{8} \end{array}\right) R_3 \rightarrow -\frac{1}{8}R_3$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{4} & -\frac{3}{8} & \frac{3}{8} \\ 0 & 1 & 0 & \frac{1}{4} & \frac{1}{8} & -\frac{1}{8} \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{8} & \frac{9}{8} \end{array}\right) \begin{array}{l} R_1 \rightarrow R_1 + 3R_3 \\ R_2 \rightarrow R_2 - R_3 \end{array} \quad \Rightarrow A^{-1} = \begin{pmatrix} \frac{1}{4} & -\frac{3}{8} & \frac{3}{8} \\ \frac{1}{4} & \frac{1}{8} & -\frac{1}{8} \\ -\frac{1}{4} & -\frac{1}{8} & \frac{9}{8} \end{pmatrix}$$

$$A^{-1} = \frac{1}{8}\begin{pmatrix} 2 & -3 & 3 \\ 2 & 1 & -1 \\ -2 & -1 & 9 \end{pmatrix}$$

12. Solve the following system of linear equation by Cramer's rule method

$\mathbf{2y = 2 + 4x + 9z,\ \ y = \frac{1}{4}[5 - z - 3x],\ \ x - 8 = 3y - 2z}$

$2y = 2 + 4x + 9z$

$\Rightarrow -4x + 2y - 9z = 2$

$y = \frac{1}{4}[5 - z - 3x]$

$\Rightarrow 4y = 5 - z - 3x$

$\Rightarrow 3x + 4y + z = 5$

$x - 8 = 3y - 2z$

$\Rightarrow x - 3y + 2z = 8$

Thus, the given system is,

$-4x + 2y - 9z = 2$

$3x + 4y + z = 5$

$x - 3y + 2z = 8$

$$\Delta = \begin{vmatrix} -4 & 2 & -9 \\ 3 & 4 & 1 \\ 1 & -3 & 2 \end{vmatrix} = -4(8+3) - 2(6-1) - 9(-9-4)$$

$$= -4(11) - 2(5) - 9(-13)$$

$$\Delta = -44 - 10 + 117 = 63$$

$$\Delta_1 = \begin{vmatrix} 2 & 2 & -9 \\ 5 & 4 & 1 \\ 8 & -3 & 2 \end{vmatrix} = 2(8+3) - 2(10-8) - 9(-15-32)$$

$$= 22 - 4 + 423$$

$$\Delta_1 = 441$$

$$\Delta_2 = \begin{vmatrix} -4 & 2 & -9 \\ 3 & 5 & 1 \\ 1 & 8 & 2 \end{vmatrix} = -4(10-8) - 2(6-1) - 9(24-5)$$

$$= -8 - 10 - 171$$

$$\Delta_2 = -189$$

$$\Delta_3 = \begin{vmatrix} -4 & 2 & 2 \\ 3 & 4 & 5 \\ 1 & -3 & 8 \end{vmatrix} = -4(32+15) - 2(24-5) + 2(-9-4)$$

$$= -4(47) - 2(19) + 2(-13)$$

$$= -188 - 38 - 26$$

$$\Delta_3 = -252$$

By Cramer's rule,

$$x = \frac{\Delta_1}{\Delta} = \frac{441}{63} = 7$$

$$y = \frac{\Delta_2}{\Delta} = -\frac{189}{63} = -3$$

$$z = \frac{\Delta_3}{\Delta} = -\frac{252}{63} = -4$$

The solution is $x = 7, y = -3, z = -4$

13. Examine the consistency of the following system of equations. If it is consistent then solve them $x + 4y - 2z = 3$, $3x + y + 5z = 7$, $2x + 3y + z = 5$

The number of unknowns = 3

$AX = B$, where

$$A = \begin{bmatrix} 1 & 4 & -2 \\ 3 & 1 & 5 \\ 2 & 3 & 1 \end{bmatrix}, \; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \; B = \begin{bmatrix} 3 \\ 7 \\ 5 \end{bmatrix}$$

The augmented matrix is

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 4 & -2 & 3 \\ 3 & 1 & 5 & 7 \\ 2 & 3 & 1 & 5 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 4 & -2 & 3 \\ 0 & -11 & 11 & -2 \\ 0 & -5 & 5 & -1 \end{array}\right] \begin{array}{l} R_2 \rightarrow R_2 - 3R_1 \\ R_3 \rightarrow R_3 - 2R_1 \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 4 & -2 & 3 \\ 0 & -1 & 1 & -\frac{2}{11} \\ 0 & -1 & 1 & -\frac{1}{5} \end{array}\right] \begin{array}{l} R_2 \rightarrow \frac{1}{11} R_2 \\ R_3 \rightarrow \frac{1}{5} R_3 \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 4 & -2 & 3 \\ 0 & -1 & 1 & -\frac{2}{11} \\ 0 & 0 & 0 & -\frac{1}{55} \end{array}\right] R_3 \rightarrow R_3 - R_2$$

The last equivalent matrix is in row - echelon form. $\rho(A) = 2,\; \rho([A|B]) = 3$

Since $\rho(A) \neq \rho([A|B])$, the given system is inconsistent and has no solution.

5 Marks

1. **Examine the consistency of the system of equations $4x + 3y + 6z = 25, x + 5y + 7z = 13$, $2x + 9y + z = 1$. If it is consistent then solve.**

$4x + 3y + 6z = 25, x + 5y + 7z = 13, 2x + 9y + z = 1$ PTA-1

$$A = \begin{bmatrix} 4 & 3 & 6 \\ 1 & 5 & 7 \\ 2 & 9 & 1 \end{bmatrix}, B = \begin{bmatrix} 25 \\ 13 \\ 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$[A|B] = \begin{bmatrix} 4 & 3 & 6 & 25 \\ 1 & 5 & 7 & 13 \\ 2 & 9 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 5 & 7 & 13 \\ 4 & 3 & 6 & 25 \\ 2 & 9 & 1 & 1 \end{bmatrix} R_1 \leftrightarrow R_2$$

$$= \begin{bmatrix} 1 & 5 & 7 & 13 \\ 0 & -17 & -22 & 27 \\ 0 & -1 & -13 & 25 \end{bmatrix} \begin{matrix} R_2 \to -R_2 \\ R_3 \to -R_3 \end{matrix}$$

$$= \begin{bmatrix} 1 & 5 & 7 & 13 \\ 0 & 17 & 22 & 27 \\ 0 & 0 & 199 & 398 \end{bmatrix} R_3 \to 17R_3 - R_2$$

$\rho(A) = 3, \rho(A/B) = 3$

$\rho(A) = \rho(A/B) = 3 =$ No. of unknowns

The system is consistent and has unique solution

$x + 5y + 7z = 13$ .... (1)

$17y + 22z = 27$ .... (2)

$199z = 398$ .... (3)

$z = \frac{398}{199} = 2$

Sub $z = 2$ in (2)

$17y + 22(2) = 27$

$17y + 44 = 27$

$17y = -17$

$y = -1$

Sub $y = -1, z = 2$ in (1)

$x + 5(-1) + 7(2) = 13$

$x - 5 + 14 = 13$

$x = 4$

Solution: $x = 4, y = -1, z = 2$

2. Decrypt the received message $[23 \quad -35 \quad 18][79 \quad -56 \quad 60][14 \quad -5 \quad 8]$ with the encryption matrix $\begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & 2 \\ 1 & -2 & 1 \end{bmatrix}$

Let the encoding matrix be $A = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & 2 \\ 1 & -2 & 1 \end{bmatrix}$

$$|A| = 2[0+4] + 1[1-2] + 0 = 8 - 1 = 7$$

$$adj\ A = \begin{bmatrix} \begin{vmatrix} 0 & 2 \\ -2 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 1 & -2 \end{vmatrix} \\ -\begin{vmatrix} -1 & 0 \\ -2 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & -1 \\ 1 & -2 \end{vmatrix} \\ \begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 5 \\ 3 & 1 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} (0+4) & -(1-2) & (-2-0) \\ -(-1+0) & (2-0) & -(-4+1) \\ (-2-0) & -(4-0) & (0+1) \end{bmatrix}^T$$

$$= \begin{bmatrix} 4 & 1 & -2 \\ 1 & 2 & 3 \\ -2 & -4 & 1 \end{bmatrix}^T = \begin{bmatrix} 4 & 1 & -2 \\ 1 & 2 & -4 \\ -2 & 3 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\, adj\ A$$

$$= \frac{1}{7}\begin{bmatrix} 4 & 1 & -2 \\ 1 & 2 & -4 \\ -2 & 3 & 1 \end{bmatrix}$$

Coded row matrix   Decoding matrix   Decoded row matrix

$$[23 \quad -35 \quad 18]\ \frac{1}{7}\begin{bmatrix} 4 & 1 & -2 \\ 1 & 2 & -4 \\ -2 & 3 & 1 \end{bmatrix} = \frac{1}{7}[92-35-36 \quad 23-70+54 \quad -46+140+18]$$

$$= \frac{1}{7}[21 \quad 7 \quad 112]$$

$$= [3 \quad 1 \quad 16] = [C \quad A \quad P]$$

$$[79 \quad -56 \quad 60]\ \frac{1}{7}\begin{bmatrix} 4 & 1 & -2 \\ 1 & 2 & -4 \\ -2 & 3 & 1 \end{bmatrix}$$

$$= \frac{1}{7}[316-56-120 \quad 79-112+180 \quad -158+224+60]$$

$$= \frac{1}{7}[140 \quad 147 \quad 126]$$

$$= [20 \quad 21 \quad 18] = [T \quad U \quad R]$$

$$[14 \quad -5 \quad 8]\ \frac{1}{7}\begin{bmatrix} 4 & 1 & -2 \\ 1 & 2 & -4 \\ -2 & 3 & 1 \end{bmatrix} = \frac{1}{7}[56-5-16 \quad 14-10+24 \quad -28+20+8]$$

$$= \frac{1}{7}[35 \quad 28 \quad 0]$$

$$= [5 \quad 4 \quad 0] = [E \quad D \quad \ ]$$

Received Message is “ Captured”

3. **Find the values of a, b, c if $A = \begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}$ is orthogonal. Hence find A^{-1}**

$$A = \begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix},$$

If A is orthogonal, $AA^T = A^TA = I$

$$AA^T = I$$

$$\begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}\begin{bmatrix} 0 & a & a \\ 2b & b & -b \\ c & -c & c \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0+4b^2+c^2 & 0+2b^2-c^2 & 0-2b^2+c^2 \\ 0+2b^2-c^2 & a^2+b^2+c^2 & a^2-b^2-c^2 \\ 0-2b^2+c^2 & a^2-b^2-c^2 & a^2+b^2+c^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$4b^2 + c^2 = 1$ .... (1)

$2b^2 - c^2 = 0$ .... (2)

$a^2 + b^2 + c^2 = 1$ .... (3)

$a^2 - b^2 - c^2 = 0$ .... (4)

$(3) + (4) \Rightarrow 2a^2 = 1 \Rightarrow a^2 = \frac{1}{2}$

$\Rightarrow a = \pm\frac{1}{\sqrt{2}}$

$(3) - (4) \Rightarrow 2b^2 + 2c^2 = 1$ ....(5)

$(5) - (2) \Rightarrow 3c^2 = 1 \Rightarrow c^2 = \frac{1}{3}$

$\Rightarrow c = \pm\frac{1}{\sqrt{3}}$

$(2) \Rightarrow 2b^2 = c^2 = \frac{1}{3} \Rightarrow b^2 = \frac{1}{6}$

$\Rightarrow b = \pm\frac{1}{\sqrt{6}}$

$\therefore a = \pm\frac{1}{\sqrt{2}}\ ,\ b = \pm\frac{1}{\sqrt{6}}\ ,\ c = \pm\frac{1}{\sqrt{3}}$

$AA^T = I$

$(A^{-1}A)A^T = A^{-1}I$

$IA^T = A^{-1}$

$A^{-1} = A^T$

$$A^{-1} = \begin{bmatrix} 0 & a & a \\ 2b & b & -b \\ c & -c & c \end{bmatrix} = \pm\begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$A^{-1} = \pm\sqrt{6}\begin{bmatrix} 0 & \sqrt{3} & \sqrt{3} \\ 2 & 1 & -1 \\ \sqrt{2} & -\sqrt{2} & \sqrt{2} \end{bmatrix}$$

4. The cost of 4kg onion, 3 kg wheat and 2kg rice is ₹ 60. The cost of 2 kg onion, 4 kg wheat and 6kg rice is ₹ 90. The cost of 6kg onion 2kg wheat and 3kg rice is ₹ 70. Find the cost of each item per kg by matrix inversion method.

Let the cost of 1 kg onion = x ; 1 kg wheat = y ; 1 kg rice = z

According to the given information, we have

$$4x + 3y + 2z = 60$$
$$2x + 4y + 6z = 90$$
$$6x + 2y + 3z = 70$$

Matrix form is $AX = B$, where

$$A = \begin{bmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{bmatrix},\ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix},\ B = \begin{bmatrix} 60 \\ 90 \\ 70 \end{bmatrix}$$

$$|A| = 4[12 - 12] - 3[6 - 36] + 2[4 - 24] = 0 + 90 - 40 = 50$$

$|A| \neq 0 \qquad \therefore A^{-1}$ exists.

$$adj\ A = \begin{bmatrix} (12-12) & -(6-36) & (4-24) \\ -(9-4) & (12-12) & -(8-18) \\ (18-8) & -(24-4) & (16-6) \end{bmatrix}^T$$

$$= \begin{bmatrix} 0 & 30 & -20 \\ -5 & 0 & 10 \\ 10 & -20 & 10 \end{bmatrix}^T = \begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix}$$

$$adj\ A = 5\begin{bmatrix} 0 & -1 & 2 \\ 6 & 0 & -4 \\ -4 & 2 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\ adj\ A = \frac{1}{50} \times 5\begin{bmatrix} 0 & -1 & 2 \\ 6 & 0 & -4 \\ -4 & 2 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{10}\begin{bmatrix} 0 & -1 & 2 \\ 6 & 0 & -4 \\ -4 & 2 & 2 \end{bmatrix}$$

$$AX = B$$

$$X = A^{-1}B = \frac{1}{10}\begin{bmatrix} 0 & -1 & 2 \\ 6 & 0 & -4 \\ -4 & 2 & 2 \end{bmatrix}\begin{bmatrix} 60 \\ 90 \\ 70 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 2 \\ 6 & 0 & -4 \\ -4 & 2 & 2 \end{bmatrix}\begin{bmatrix} 6 \\ 9 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 - 9 + 14 \\ 36 + 0 - 28 \\ -24 + 18 + 14 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 8 \end{bmatrix}$$

$x = 5, y = 8, z = 8$

The cost of 1 kg onion = ₹ 5

The cost of 1 kg wheat = ₹ 8

The cost of 1kg rice = ₹ 8

5. **There were 240 persons in a picnic. There were 20 more men than women and 20 more adults than children. How many men and women and children were there in the picnic?**

Let the number of men $= x$; The number of women $= y$; The number of children $= z$

$\therefore$ Number of adults $= x + y$

According to the conditions, we get,

$$x + y + z = 240$$
$$x - y = 20$$
$$x + y - z = 20$$

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{vmatrix} = 1(1-0) - 1(-1-0) + 1(1+1) = 1 + 1 + 2 = 4$$

$$\Delta_1 = \begin{vmatrix} 240 & 1 & 1 \\ 20 & -1 & 0 \\ 20 & 1 & -1 \end{vmatrix} = 240(1-0) - 1(-20-0) + 1(20+20) = 240 + 20 + 40 = 300$$

$$\Delta_2 = \begin{vmatrix} 1 & 240 & 1 \\ 1 & 20 & 0 \\ 1 & 20 & -1 \end{vmatrix} = 1(-20-0) - 240(-1-0) + 1(20-20) = -20 + 240 + 0 = 220$$

$$\Delta_3 = \begin{vmatrix} 1 & 1 & 240 \\ 1 & -1 & 20 \\ 1 & 1 & 20 \end{vmatrix} = 1(-20-20) - 1(20-20) + 240(1+1) = -40 - 0 + 480 = 440$$

By Cramer's rule, $x = \frac{\Delta_1}{\Delta} = \frac{300}{4} = 75$

$y = \frac{\Delta_2}{\Delta} = \frac{220}{4} = 55$

$z = \frac{\Delta_3}{\Delta} = \frac{440}{4} = 110$

$\therefore$ In the picnic, there were 75 men, 55 women, and 110 children.

6. ***A*, *B* and *C* work in telemarketers. Between the three of them, they can process 570 orders in a day. A process 60 more orders in one day than *B*. *C* process 30 less orders in one day than *A*. How many orders in one day does each of these individuals process?**

Let the number of orders processed by $A = x$

The number of orders processed by $B = y$

The number of orders processed by $C = z$

Given: Total orders processed in a day by all the three =570

$$x + y + z = 570$$

Given: Number of orders processed by A in one day

= {No. of orders processed by B in one day} + 60

$$x = y + 60$$
$$x - y = 60$$

Given: Number of orders processed by C in one day

= {No. of orders processed by A in one day} -30

$$z = x - 30$$
$$x - z = 30$$

The system of linear equations: $x + y + z = 570$

$$x - y = 60$$
$$x - z = 30$$

The augmented matrix is $(A|B) = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 570 \\ 1 & -1 & 0 & 60 \\ 1 & 0 & -1 & 30 \end{array}\right)$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 570 \\ 0 & -2 & -1 & -510 \\ 0 & -1 & -2 & -540 \end{array}\right) \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 570 \\ 0 & 2 & 1 & 510 \\ 0 & 1 & 2 & 540 \end{array}\right) \begin{array}{l} R_2 \to -R_2 \\ R_3 \to -R_3 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 570 \\ 0 & 2 & 1 & 510 \\ 0 & 0 & 3 & 570 \end{array}\right) R_3 \to 2R_3 - R_2$$

Writing the equivalent system of equation from the row-echelon matrix, we get

$x + y + z = 570$ ……………… (1)

$2y + z = 510$ ……………… (2)

$3z = 570$ ……………… (3)

$(3) \Rightarrow 3z = 570$

$z = \frac{570}{3} = 190$

$(2) \Rightarrow 2y + 190 = 510$

$2y = 510 - 190 = 320$

$y = \frac{320}{2} = 160$

$(1) \Rightarrow x + 160 + 190 = 570$

$x + 350 = 570$

$x = 570 - 350 = 220$

In one day, The number of orders processed by $A = 220$

The number of orders processed by $B = 160$

The number of orders processed by $C = 190$

7. Find a, b, c when $f(x) = ax^2 + bx + c$, $f(0) = 6$, $f(2) = 11$ and $f(-3) = 6$. Determine the quadratic function $f(x)$ and find its value when $x = 1$

$f(x) = ax^2 + bx + c$

$f(0) = a(0^2) + b(0) + c$

$c = 6$

$f(2) = a(2^2) + b(2) + c = 11$

$4a + 2b + c = 11$

$f(-3) = a(-3)^2 + b(-3) + c = 6$

$9a - 3b + c = 6$

The augmented matrix is

$$(A|B) = \left(\begin{array}{ccc|c} 0 & 0 & 1 & 6 \\ 4 & 2 & 1 & 11 \\ 9 & -3 & 1 & 6 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 0 & 0 & 1 & 6 \\ 4 & 2 & 1 & 11 \\ 0 & -30 & -5 & -75 \end{array}\right) R_3 \rightarrow 4R_3 - 9R_2$$

$$\rightarrow \left(\begin{array}{ccc|c} 4 & 2 & 1 & 11 \\ 0 & 0 & 1 & 6 \\ 0 & 6 & 1 & 15 \end{array}\right) \begin{array}{l} R_1 \leftrightarrow R_2 \\ R_3 \rightarrow \frac{1}{-5} R_3 \end{array}$$

$$\rightarrow \left(\begin{array}{ccc|c} 4 & 2 & 1 & 11 \\ 0 & 6 & 1 & 15 \\ 0 & 0 & 1 & 6 \end{array}\right) R_2 \leftrightarrow R_3$$

Writing the equivalent system of equations from the row-echelon matrix, we get,

$4a + 2b + c = 11$ .... (1)

$6b + c = 15$ .... (2)

$c = 6$ .... (3)

$(3) \Rightarrow c = 6$

$(2) \Rightarrow 6b + 6 = 15$

$6b = 15 - 6$

$b = \frac{9}{6} = \frac{3}{2}$

$(1) \Rightarrow 4a + 2\left(\frac{3}{2}\right) + 6 = 11$

$4a + 9 = 11$

$4a = 11 - 9 = 2$

$a = \frac{1}{2}$

The required quadratic equation is $f(x) = \frac{1}{2}x^2 + \frac{3}{2}x + 6$

When $x = 1$,

$f(1) = \frac{1}{2}(1)^2 + \frac{3}{2}(1) + 6$

$= \frac{1}{2} + \frac{3}{2} + 6$

$f(1) = 8$

8. Investigate for what values of m and n the equations $x + y + 2z = 2$, $2x - y + 3z = 2$, $5x - y + mz = n$ have (i) no solution (ii) unique solution (iii) infinite number of solutions.

The matrix form of the given system of equations is $AX = B$, where

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 3 \\ 5 & -1 & m \end{bmatrix}, \; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \; B = \begin{bmatrix} 2 \\ 2 \\ n \end{bmatrix}$$

The augmented matrix is

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 2 & -1 & 3 & 2 \\ 5 & -1 & m & n \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & -3 & -1 & -2 \\ 0 & -6 & m-10 & n-10 \end{array}\right] \begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - 5R_1 \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & -3 & -1 & -2 \\ 0 & 0 & m-8 & n-6 \end{array}\right] R_3 \rightarrow R_3 - 2R_2$$

(i) If $m = 8$ and $n \neq 6$,

$\rho(A) = 2$ and $\rho([A|B]) = 3$

Since $\rho(A) \neq \rho([A|B])$, the given system has no solution.

(ii) If $m \neq 8, n \in R$

$\rho(A) = \rho([A|B]) = 3 =$ No. of unknowns.

$\therefore$ The given system has unique solution.

(iii) If $m = 8$ and $n = 6$,

$\rho(A) = \rho([A|B]) = 2 <$ no. of unknowns.

$\therefore$ The given system has infinite number of solutions.

(i) The given system has no solution for $m = 8$ and $n \neq 6$

(ii) The given system has unique solution for $m \neq 8$ and $n \in R$

(iii) The given system has infinite number of solutions for $m = 8$ and $n = 6$

9. By using Gaussian elimination method balance the chemical reaction equation.
$\boldsymbol{BF_3 + NaH \rightarrow B_2H_6 + 6NaF}$

We are searching for positive integers x_1, x_2, x_3 and x_4 such that

$$x_1BF_3 + x_2NaH \rightarrow x_3B_2H_6 + 6x_4NaF \quad \text{.............. (1)}$$

The number of Boron (B) atoms on the left-hand side of (1) should be equal to the number of Boron atoms on the right-hand side of (1).

$$x_1 = 2x_3 \Rightarrow x_1 - 2x_3 = 0 \quad \text{.............. (2)}$$

Similarly, considering Fluorine, Sodium and Hydrogen atoms, we get respectively,

$$3x_1 = 6x_4 \Rightarrow x_1 = 2x_4 \Rightarrow x_1 - 2x_4 = 0 \quad \text{.............. (3)}$$
$$x_2 = 6x_4 \Rightarrow x_2 - 6x_4 = 0 \quad \text{.............. (4)}$$
$$x_2 = 6x_3 \Rightarrow x_2 - 6x_3 = 0 \quad \text{.............. (5)}$$

Equations (2), (3), (4) and (5) constitute a homogeneous system of linear equation in four unknowns.

The augmented

$$[A|B] = \left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 0 \\ 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & -6 & 0 \\ 0 & 1 & -6 & 0 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 0 \\ 0 & 0 & 2 & -2 & 0 \\ 0 & 1 & 0 & -6 & 0 \\ 0 & 0 & -6 & 6 & 0 \end{array}\right] \begin{array}{l} R_2 \rightarrow R_2 - R_1 \\ \\ R_4 \rightarrow R_4 - R_3 \end{array}$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & 0 & -6 & 0 \\ 0 & 0 & 2 & -2 & 0 \\ 0 & 0 & -6 & 6 & 0 \end{array}\right] R_2 \leftrightarrow R_3$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & 0 & -6 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 \end{array}\right] \begin{array}{l} R_3 \rightarrow \frac{1}{2} R_3 \\ R_4 \rightarrow \frac{1}{6} R_4 \end{array}$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & 0 & -6 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] R_4 \rightarrow R_4 + R_3$$

$\therefore \rho(A) = \rho([A|B]) = 3 < 4 =$ Number of unknowns.

$\therefore$ The system is consistent and has infinite number of solutions.

Equivalent system of equations from the row-echelon matrix is

$$x_1 - 2x_3 = 0 \quad \text{.............. (6)}$$
$$x_2 - 6x_4 = 0 \quad \text{.............. (7)}$$
$$x_3 - x_4 = 0 \quad \text{................. (8)}$$

Let $x_4 = t,\ t \in R - \{0\}$

$(8) \Rightarrow x_3 - t = 0 \Rightarrow x_3 = t$

$(7) \Rightarrow x_2 - 6t = 0 \Rightarrow x_2 = 6t$

$(8) \Rightarrow x_1 - 2t = 0 \Rightarrow x_1 = 2t$

$(x_1, x_2, x_3, x_4) = (2t, 6t, t, t)$

Let us choose $t = 1$

Then, we get $x_1 = 2(1) = 2$, $x_2 = 6(1) = 6$, $x_3 = 1$ and $x_4 = 1$

So, the balanced equation is

$$2BF_3 + 6NaH \rightarrow B_2H_6 + 6NaH$$

Important Example Questions:
2 Marks: Eg.1.11 (PTA-1)
3 Marks: Eg.1.8 (Mar-20)
5 Marks: Eg.1.34 (PTA-2), Eg.1.21 (PTA-6), Eg.1.32 (Mar-20)

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