4. Geometry

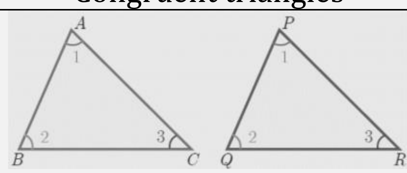
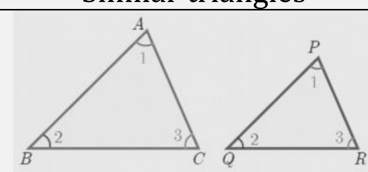
Introduction for Exercise 4.1

Concept corner

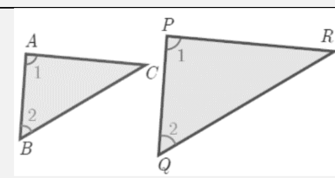
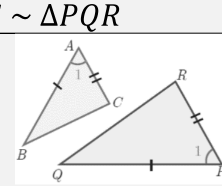
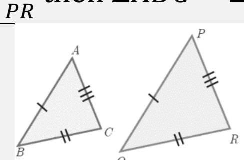
- Two figures are said to be similar if every aspect of one figure is proportional to other figure.

Congruency and similarity of triangles

- Congruency is a particular case of similarity. In both the cases, three angles of one triangle are equal to the three corresponding angles of the other triangle.
- But in congruent triangles, the corresponding sides are equal. While in similar triangles, the corresponding sides are proportional.
- The triangles ABC and PQR are similar can be written as $\Delta ABC \sim \Delta PQR$

Congruent triangles	Similar triangles
 $\Delta ABC \cong \Delta PQR$ $\angle A = \angle P, \angle B = \angle Q, \angle C = \angle R$ $AB = PQ, BC = QR, CA = RP$ $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{CA}{RP} = 1$ Same shape and same size	 $\Delta ABC \sim \Delta PQR$ $\angle A = \angle P, \angle B = \angle Q, \angle C = \angle R$ $AB \neq PQ, BC \neq QR, CA \neq RP$ $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{CA}{RP} > 1$ or < 1 Same shape but not same size

Criteria of Similarity

AA Criterion of similarity	If two angles of one triangle are respectively equal to two angles of another triangle, then the two triangles are similar, because the third angle in both triangles must be equal. Therefore, AA similarity criterion is same as the AAA similarity criterion	 So, if $\angle A = \angle P = 1$ and $\angle B = \angle Q = 2$ then $\Delta ABC \sim \Delta PQR$
SAS Criterion of similarity	If one angle of a triangle is equal to one angle of another triangle and if the sides including them are proportional then the two triangles are similar.	 Thus, if $\angle A = \angle P = 1$ and $\frac{AB}{PQ} = \frac{AC}{PR}$ then $\Delta ABC \sim \Delta PQR$
SSS Criterion of similarity	If three sides of a triangle are proportional to the three corresponding sides of another triangle, then the two triangles are similar.	 So if, $\frac{AB}{PQ} = \frac{AC}{PR} = \frac{BC}{QR}$ then $\Delta ABC \sim \Delta PQR$

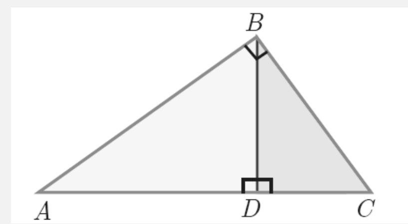
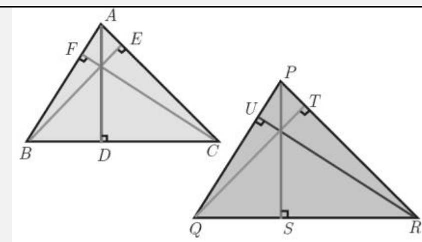
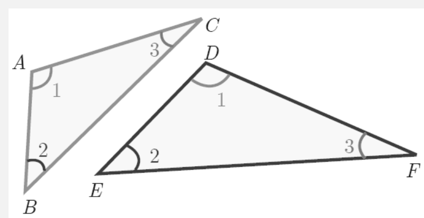
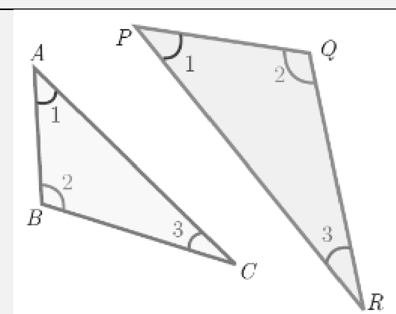
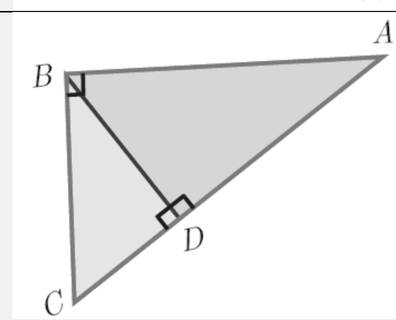
Definition:

1. Two triangles are said to be similar if their corresponding sides are proportional.
2. The triangles are equiangular if the corresponding angles are equal.

Note:

- If we change exactly one of the four given lengths, then we can make these triangles are similar
- A pair of equiangular triangles are similar.
- If two triangles are similar, then they are equiangular.

Some useful results on Similar Triangles:

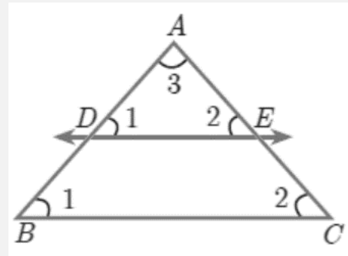
1	A perpendicular line drawn from the vertex of a right angled triangle divides the triangle into two triangles similar to each other and also to original triangle. $\Delta ADB \sim \Delta BDC$, $\Delta ABC \sim \Delta ADB$, $\Delta ABC \sim \Delta BDC$	
2	If two triangles are similar, then the ratio of the corresponding sides are equal to the ratio of their corresponding altitudes. If $\Delta ABC \sim \Delta PQR$ then $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{CA}{RP} = \frac{AD}{PS} = \frac{BE}{QT} = \frac{CF}{RU}$	
3	If two triangles are similar, then the ratio of the corresponding sides are equal to the ratio of the corresponding perimeters. $\Delta ABC \sim \Delta DEF$ then $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD} = \frac{AB+BC+CA}{DE+EF+FD}$	
4	The ratio of the area of two similar triangles are equal to the ratio of the squares of their corresponding sides $\frac{area\ (\Delta ABC)}{area\ (\Delta PQR)} = \frac{AB^2}{PQ^2} = \frac{BC^2}{QR^2} = \frac{AC^2}{PR^2}$	
5	If two triangles have common vertex and their bases are on the same straight line, the ratio between their areas is equal to the ratio between the length of their bases. $\frac{area\ (\Delta ABD)}{area\ (\Delta BDC)} = \frac{AD}{DC}$	

Introduction for Exercise 4.2

Concept corner

Theorem 1: Basic Proportionality Theorem (BPT) or Thales theorem

Statement: A straight line drawn parallel to a side of triangle intersecting the other two sides, divides the sides in the same ratio.



Proof:

Given: In ΔABC, D is a point on AB and E is a point on AC.

To prove: $\frac{AD}{DB} = \frac{AE}{EC}$

Construction: Draw a line $DE \parallel BC$

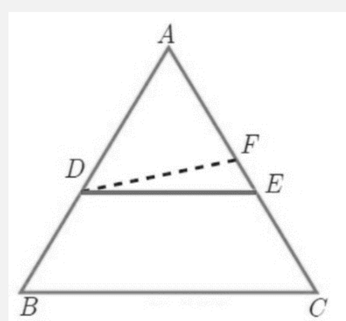
No.	Statement	Reason
1.	$\angle ABC = \angle ADE = \angle 1$	Corresponding angles are equal because $DE \parallel BC$
2.	$\angle ACB = \angle AED = \angle 2$	Corresponding angles are equal because $DE \parallel BC$
3.	$\angle DAE = \angle BAC = \angle 3$	Both triangles have a common angle
4.	$\Delta ABC \backsim \Delta ADE$	By AAA similarity
	$\frac{AB}{AD} = \frac{AC}{AE}$	Corresponding sides are proportional
	$\frac{AD+DB}{AD} = \frac{AE+EC}{AE}$	Split AB and AC using the points D and E
	$1 + \frac{DB}{AD} = 1 + \frac{EC}{AE}$	On simplification
	$\frac{DB}{AD} = \frac{EC}{AE}$	Cancelling 1 on both sides
	$\frac{AD}{DB} = \frac{AE}{EC}$	Taking reciprocals
	Hence proved	

Corollary: If in ΔABC, a straight line DE parallel to BC, intersects AB at D and AC at E, then

(i) $\frac{AB}{AD} = \frac{AC}{AE}$ (ii) $\frac{AB}{DB} = \frac{AC}{EC}$

Theorem 2: Converse of Basic Proportionality Theorem

Statement: If a straight line divides any two sides of a triangle in the same ratio, then the line must be parallel to the third side.



Proof:

Given: In ΔABC, $\frac{AD}{DB} = \frac{AE}{EC}$

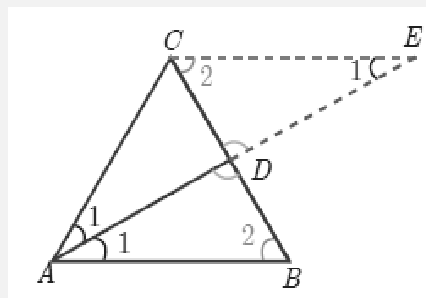
To prove: $DE \parallel BC$

Construction: Draw $BF \parallel DE$

No.	Statement	Reason
1.	In $\Delta ABC,\ BF \parallel DE$	Construction
2.	$\frac{AD}{DB} = \frac{AE}{EC}$ ……….. (1)	Thales theorem (In ΔABC taking D in $AB$and in AC)
3.	$\frac{AD}{EC} = \frac{AF}{FC}$ ………… (2)	Thales theorem (In ΔABC taking F in AC)
4.	$\frac{AE}{EC} = \frac{AF}{FC}$ $\frac{AE}{EC} + 1 = \frac{AF}{FC} + 1$ $\frac{AE+EC}{EC} = \frac{AF+FC}{FC}$ $\frac{AC}{EC} = \frac{AC}{FC}$ $EC = FC$ Therefore, $E = F$ Thus $DE \parallel BC$	From (1) and (2) Adding 1 to both sides Cancelling AC on both sides F lies between E and C. Hence Proved

Theorem 3: Angle Bisector Theorem

Statement: The internal bisector of an angle of a triangle divides the opposite side internally in the ratio of the corresponding sides containing the angle. PTA-5



Proof:

Given : In $\Delta ABC,\ AD$ is the internal bisector

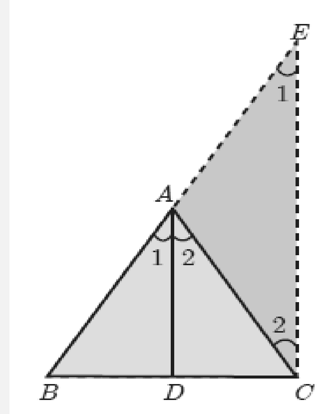
To prove: $\frac{AB}{AC} = \frac{BD}{CD}$

Construction : Draw a line through C parallel to AB. Extend AD to meet line through C at E

No.	Statement	Reason
1.	$\angle AEC = \angle BAE = \angle 1$	Two parallel lines cut by a transversal make alternate angles equal.
2.	ΔACE is isosceles $AC = CE$ …… (1)	In $\Delta ACE,\ \angle CAE = \Delta CEA$
3.	$\Delta ABD \backsim \Delta ECD$ $\frac{AB}{CE} = \frac{BD}{CD}$	By AA similarity
4.	$\frac{AB}{AC} = \frac{BD}{CD}$	From (1) $AC = CE$ Hence proved.

Theorem 4: Converse of Angle Bisector Theorem

Statement: If a straight line through one vertex of a triangle divides the opposite side internally in the ratio of the other two sides, then the line bisects the angle internally at the vertex. PTA-3, 4



Proof:

Given : ABC is a triangle.

AD divides BC in the ratio of the sides containing the angles $\angle A$ to meet BC at D.

That is $\frac{AB}{AC} = \frac{BD}{DC}$ ……… (1)

To prove : AD bisects $\angle A$ i.e. $\angle 1 = \angle 2$

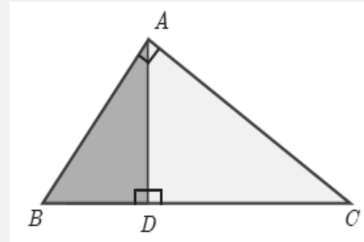
Construction : Draw $CE \parallel DA$. Extend BA to meet at E.

No.	Statement	Reason
1.	Let $\angle BAD = \angle 1$ and $\angle DAC = \angle 2$	Assumption
2.	$\angle BAD = \angle AEC = \angle 1$	Since $DA \parallel CE$ and AC is transversal, corresponding angles are equal
3.	$\angle DAC = \angle ACE = \angle 2$	Since $DA \parallel CE$ and AC is transversal, Alternate angles are equal
4.	$\frac{BA}{AE} = \frac{BD}{DC}$ ……….. (2)	In ΔBCE by thales theorem
5.	$\frac{AB}{AC} = \frac{BD}{DC}$	From (1)
6.	$\frac{AB}{AC} = \frac{BA}{AE}$	From (1) and (2)
7.	$AC = AE$ ……….(3)	Cancelling AB
8.	$\angle 1 = \angle 2$	ΔACE is isosceles by (3)
9.	AD bisects $\angle A$	Since, $\angle 1 = \angle BAD = \angle 2 = \angle DAC$. Hence proved

Note: If C_1 , C_2, … are points on the circle, then all the triangles ΔBAC_1 , ΔBAC_2 , … are with same base and the same vertical angle.

Introduction for Exercise 4.3

Concept corner



Theorem 5: Pythagoras Theorem

Statement: In a right angle triangle, the square on the hypotenuse is equal to the sum of the squares on the other two sides.

Proof: PTA-4

Given : In ΔABC, $\angle A = 90°$

To prove : $AB^2 + AC^2 = BC^2$

Construction : Draw $AD \perp BC$

No.	Statement	Reason
1.	Compare ΔABC and ΔABD $\angle B$ is common $\angle BAC = \angle BDA = 90°$ Therefore, $\Delta ABC \backsim \Delta ABD$ $\frac{AB}{BD} = \frac{BC}{AB}$ $AB^2 = BC \times BD$ ...(1)	Given $\angle BAC = 90°$ and by construction $\angle BDA = 90°$ By AA similarity
2.	Compare ΔABC and ΔADC $\angle C$ is common $\angle BAC = \angle ADC = 90°$ Therefore, $\Delta ABC \backsim \Delta ADC$ $\frac{BC}{AC} = \frac{AC}{DC}$ $AC^2 = BC \times DC$ ... (2)	Given $\angle BAC = 90°$ and by construction $\angle CDA = 90°$ By AA similarity

Adding (1) and (2) we get

$AB^2 + AC^2 = BC \times BD + BC \times DC$

$= BC \times (BD + DC)$

$= BC \times BC$

$AB^2 + AC^2 = BC^2$

Hence the theorem is proved.

Converse of Pythagoras Theorem

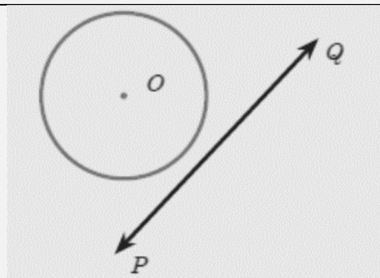
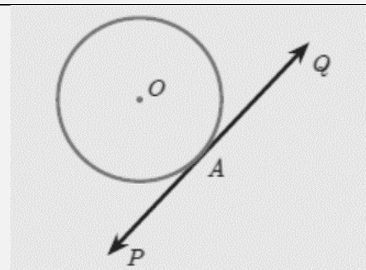
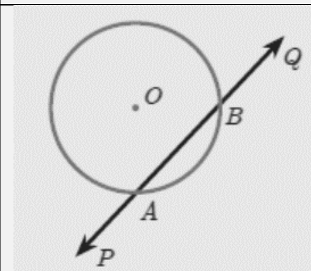
Statement: If the square of the longest side of a triangle is equal to sums of squares of other two sides, then the triangle is a right angle triangle.

Note:

- In a right angles triangle, the side opposite to 90° (the right angle) is called the hypotenuse.
- The other two sides are called legs of the right angled triangle.
- The hypotenuse will be the longest side of the triangle.

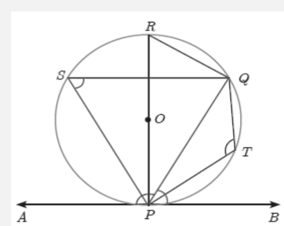
Introduction for Exercise 4.4

Concept corner

	Figure 1	Figure 2	Figure 3
			
(i)	Straight line PQ does not touch the circle.	Straight line PQ touches the circle at a common point A	Straight line PQ intersects the circle at two points A and B.
(ii)	There is no common point between the **straight line** and circle	PQ is called the **tangent** to the circle at A	The line PQ is called a **secant** of the circle
(iii)	Thus the number of **point of intersection** of a line and circle is **zero.**	Thus the number of **points of intersection** of a line and circle is **one.**	Thus the number of **points of intersection** of a line and circle is **two**

Definition: If a line touches the given circle at only one point then it is called **tangent to the circle.**

Theorem 6: Alternate Segment theorem



Statement: If a line touches a circle and from the point of contact a chord is drawn, the angles between the tangent and the chord are respectively equal to the angles in the corresponding alternate segments.

Proof:

Given : A circle with centre at O, tangent AB touches the circle at P and PQ is a chord. S and T are two points on the circle in the opposite sides of chord PQ.

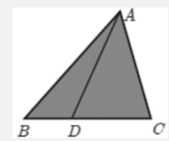
To prove : (i) $\angle QPB = \angle PSQ$ and (ii) $\angle QPA = \angle PTQ$

Construction : Draw the diameter POR. Draw QR, QS and PS.

No.	Statement	Reason
1.	$\angle RPB = 90°$ Now, $\angle RPQ + \angle QPB = 90°$ ... (1)	Diameter RP is perpendicular to tangent AB.
2.	In $\Delta RPQ, \angle PQR = 90°$ ... (2)	Angle in a semicircle is $90°$.

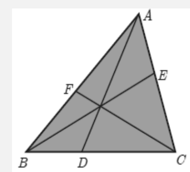
3.	$\angle QRP + \angle RPQ = 90°$ ... (3)	In a right angled triangle, sum of the two acute angles is 90°.
4.	$\angle RPQ + \angle QPB = \angle QRP + \angle RPQ$ $\angle QPB = \angle QRP$ ... (4)	From (1) and (3).
5.	$\angle QRP = \angle PSQ$ ... (5)	Angles in the same segment are equal.
6.	$\angle QPB = \angle PSQ$ ... (6)	From (4) and (5); Hence (i) is proved.
7.	$\angle QPB + \angle QPA = 180°$ ... (7)	Linear pair of angles.
8.	$\angle PSQ + \angle PTQ = 180°$ ...(8)	Sum of opposite angles of a cyclic quadrilateral is 180°.
9.	$\angle QPB + \angle QPA = \angle PSQ + \angle PTQ$	From (7) and (8).
10.	$\angle QPB + \angle QPA = \angle QPB + \angle PTQ$	$\angle QPB = \angle PSQ$ from (6)
11.	$\angle QPA = \angle PTQ$	Hence (ii) is proved. This completes the proof.

Definition: A **cevian** is a line segment that extends from one vertex of a triangle to the opposite side. In the diagram, AD is a cevian, from A.



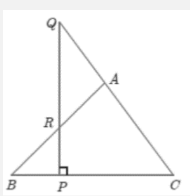
Ceva's Theorem (without proof)

Statement: Let ABC be a triangle and let D, E, F be points on lines BC, CA, AB respectively. Then the cevians AD, BE, CF are concurrent if and only if $\frac{BD}{DC} \times \frac{CE}{EA} \times \frac{AF}{FB} = 1$ where the lengths are directed. This also works for the reciprocal of each of the ratios as the reciporcal of 1 is 1.



Note: The cevians do not necessarily lie within the triangle, although they do in the diagram

Menelaus Theorem (without proof)



Statement: A necessary and sufficient condition for points P, Q, R on the respective sides BC, CA, AB (or their extension) of a triangle ABC to be collinear is that $\frac{BP}{PC} \times \frac{CQ}{QA} \times \frac{AR}{RB} = -1$ where all segments in the formula are directed segments.

Note:

- Menelaus theorem can also be given as $BP \times CQ \times AR = -PC \times QA \times RB$
- If BP is replaced by PB (or) CQ by QC (or) AR by RA, or if any one of the six directed line segments BP, PC, CQ, QA, AR, RB is interchanged, then the product will be 1.
- Centroid is the point of concurrence of the median of a triangle.