Higher Secondary 1st year
Mathematics
Departmental Model Question Paper- fully solved

Time:2 ½ hours Max. Marks: 90

Part-I **All questions are compulsory.**

Choose the correct answer.

1. Number of elements in a matrix of order 2×3 is
   a)5 b)2 c)3 **d)6**
2. If A is a square matrix of order 4 then $|KA|$ is
   a) $K|A|$ b) $K^2|A|$ c) $K^3|A|$ **d) $\boldsymbol{K^4|A|}$**
3. If "G" is centroid of the triangle ABC and 'O' is any other point then $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} =$
   a) $\vec{O}$ b) $\overrightarrow{OG}$ **c)3 $\boldsymbol{\overrightarrow{OG}}$** d) 4 $\overrightarrow{OG}$
4. If $\vec{a} = 2\vec{i} - \vec{j}$ and $\vec{b} = \vec{j} - \vec{k}$ then the magnitude of $\vec{a} - \vec{b}$ is
   a)1 b)9 **c)3** d)$\sqrt{3}$
5. A polygon has 44 diagonals, then the number of its sides is
   a)11 b) 7 c) 8 d) 12
6. If $\frac{3x+7}{(x-1)(x-2)} = \frac{A}{x-2} - \frac{10}{x-1}$ then A is
   a)13 b)-13 c) -10 d) 10
7. The A.M., G.M.,H.M between two positive numbers a and b are equal then
   a) $\boldsymbol{a = b}$ b) $ab = 1$ c) $a > b$ d) $a < b$
8. If $a_n = n^2 3^{-n}$ then the third term is
   a) $\frac{1}{9}$ b) 1 **c)$\boldsymbol{\frac{1}{3}}$** d) 3
9. The pair of straight lines given by $ax^2 + 2hxy + by^2 = 0$ are perpendicular, then
   a) $ab = 0$ **b) $\boldsymbol{a + b = 0}$** c) $a - b = 0$ d) $a = 0$
10. The radius of the circle $x^2 + y^2 - 2x + 4y - 4 = 0$ is
    a)1 b)2 **c)3** d)4
11. If $\cos\theta = 0$ then θ is
    a) $n\pi$ **b) $\boldsymbol{(2n+1)\frac{\pi}{2}}$** c)$-\pi$ d) $-n\pi$
12. If the terminal side is collinear with the initial side in the opposite direction, then the angle included is
    a) 0^0 b) 90^0 **c) $\boldsymbol{180^0}$** d) 270^0
13. The range of the function $log_e\, x$ is
    a) $(0, \infty)$ **b) $(-\infty, \infty)$** c) $(-\infty, 0)$ d) $[0, \infty)$
14. The value of the function $\lceil 3.5 \rceil$ is
    a)2 b)3 **c)4** d)5
15. $\frac{d}{dx}(\log\sqrt{x})$ is
    a)$\frac{1}{2\sqrt{x}}$ **b)$\boldsymbol{\frac{1}{2x}}$** c)$\frac{1}{x\sqrt{x}}$ d)$\frac{1}{2x\sqrt{x}}$

16. $\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^x$ is

a) e b) e^x **c) e^3** d) ∞

17. $\int \log x\, dx =$

a) $\frac{1}{x}+c$ b) $\frac{(\log x)^2}{2}+c$ c) $x\log x + x + c$ **d) $x\log x - x + c$**

18. $\int e^{2x}\sin 3x\, dx$ is

a) $\frac{e^{2x}}{13}(2\sin 3x - 3\cos 3x)+c$ b) $\frac{e^{2x}}{13}(3\sin 2x - 2\cos 2x)+c$

c) $\frac{e^{2x}}{13}(2\sin 3x + 3\cos 3x)$ d) $\frac{e^{2x}}{13}(3\cos 3x + 2\sin 3x)+c$

19. If two events A and B are independent then $P(A/B) =$ ____

a) $P(A)$ b) $P(A\cap B)$ c) $P(A)=P(B)$ d) $\frac{P(A)}{P(B)}$

20. X speaks truth in 95 of cases and Y in 80 percent of cases. The percentage of cases they likely to contradict each other in stating same fact is

a) 14% b) 86% **c) 23%** d) 85.5%

Part –II

Answer any SEVEN questions **7 x 2 = 14**

Question No. 30 is compulsory

21. Prove that $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 = \begin{vmatrix} a_1^2+a_2^2 & a_1b_1+a_2b_2 \\ a_1b_1+a_2b_2 & b_1^2+b_2^2 \end{vmatrix}$

L.H.S $= \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$

$= \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ (Interchange rows and columns of first determinant)

$= \begin{vmatrix} a_1^2+a_2^2 & a_1b_1+a_2b_2 \\ a_1b_1+a_2b_2 & b_1^2+b_2^2 \end{vmatrix}$ = R.H.S

22. If ABC and $A'B'C'$ are two triangles and G, G' be their corresponding centroids, prove that $\overrightarrow{AA}' + \overrightarrow{BB}' + \overrightarrow{CC}' = 3\overrightarrow{GG}'$

$\overrightarrow{OG} = \frac{\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}}{3}$, $\overrightarrow{OG}' = \frac{\overrightarrow{OA}'+\overrightarrow{OB}'+\overrightarrow{OC}'}{3}$

$3\overrightarrow{OG} = \overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}$, $3\overrightarrow{OG}' = \overrightarrow{OA}'+\overrightarrow{OB}'+\overrightarrow{OC}'$

$\overrightarrow{AA}' + \overrightarrow{BB}' + \overrightarrow{CC}' = \overrightarrow{OA}' - \overrightarrow{OA} + \overrightarrow{OB}' - \overrightarrow{OB} + \overrightarrow{OC}' - \overrightarrow{OC}$

$= (\overrightarrow{OA}' + \overrightarrow{OB}' + \overrightarrow{OC}') - (\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC})$

$= 3\overrightarrow{OG}' - 3\overrightarrow{OG} = 3(\overrightarrow{OG}' - \overrightarrow{OG}) = 3\overrightarrow{GG}'$

23. If $10P_r = 5040$, find the value of r

$$nP_r = \frac{n!}{(n-r)!}$$

$$\frac{10!}{(10-r)!} = 5040$$
$$\frac{10!}{5040} = (10-r)!$$
$$(10-r)! = \frac{10\times9\times8\times7\times6!}{10\times9\times8\times7}$$
$$(10-r)! = 6!$$
$$(10-r) = 6$$
$$10-6 = r$$
$$r = 4$$

24. A point moves so that it is always at a distance of 6 units from the point $(1,-4)$. Find its locus

Let $P(x_1, y_1)$ be the moving point. Its distance from $(1,-4)$ is given as 6 units.

$$\sqrt{(x_1-1)^2 + (y_1+4)^2} = 6$$

Taking square on both sides

$$(a+b)^2 = a^2+b^2+2ab$$
$$(a-b)^2 = a^2+b^2-2ab$$

$$(x_1-1)^2 + (y_1+4)^2 = 36$$
$$x_1^2 + 1^2 - 2(x_1)(1) + y_1^2 + 4^2 + 2(y_1)(4) = 36$$
$$x_1^2 + 1 - 2x_1 + y_1^2 + 16 + 8y_1 = 36$$
$$x_1^2 + y_1^2 - 2x_1 + 8y_1 = 36 - 17$$
$$x_1^2 + y_1^2 - 2x_1 + 8y_1 = -19$$

$\therefore$ locus of (x_1, y_1) is $x^2 + y^2 - 2x + 8y - 19 = 0$

25. Simplify: $\cos(-870^0)$

$$\cos(-870^0) = \cos(870^0)$$
$$= \cos\left((2\times360^0) + 150^0\right) = \cos 150^0 = \cos(180^0 - 30^0)$$
$$= -\cos 30^0 = -\frac{\sqrt{3}}{2}$$

26. If $f, g: R \to R$, defined by $f(x) = x+1$ and $g(x) = x^2$ then find $(f\circ g)(3)$

$f(x) = x+1$ and $g(x) = x^2$

$(f\circ g)(3) = f[g(3)] = f(3^2) = f(9) = 9+1=10$

27. Find $\frac{d^2y}{dx^2}$ if $y = x^3 - 6x^2 + 7x + 6$

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\frac{dy}{dx} = 3x^2 - 12x + 7$$

$$\frac{d^2y}{dx^2} = 6x - 12$$

28. Evaluate: $\int \cos^2 x\, dx$

$\cos^2 x = \frac{1+\cos 2x}{2}$

$\int \cos^2 x\, dx = \int \frac{1+\cos 2x}{2} dx$

$= \int \left(\frac{1}{2} + \frac{\cos 2x}{2}\right) dx$

$= \int \left(\frac{1}{2}\right) dx + \int \left(\frac{\cos 2x}{2}\right) dx$

$= \frac{1}{2}\int dx + \frac{1}{2}\int \cos 2x\, dx$

$= \frac{1}{2}x + \frac{\sin 2x}{2\times 2}$

$= \frac{x}{2} + \frac{\sin 2x}{4}$

29. Three coins are tossed once. Find the probability of getting atleast two heads.

Three coins are tossed. $n(S) = 8$

Probability of getting atleast two heads $= P(2) + P(3) = 3C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right) + 3C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^0$

$= \frac{3\times 2}{1\times 2}\left(\frac{1}{4}\right)\left(\frac{1}{2}\right) + 1\left(\frac{1}{8}\right) = 3\left(\frac{1}{8}\right) + \frac{1}{8} = \frac{3}{8} + \frac{1}{8} = \frac{3+1}{8} = \frac{4}{8} = \frac{1}{2}$

30. Show that : $\frac{e^2-1}{e^2+1} = \frac{\frac{1}{1!}+\frac{1}{3!}+\frac{1}{5!}+\cdots}{1+\frac{1}{2!}+\frac{1}{4!}+\cdots}$

R.H.S $= \frac{\frac{1}{1!}+\frac{1}{3!}+\frac{1}{5!}+\cdots}{1+\frac{1}{2!}+\frac{1}{4!}+\cdots} = \frac{\frac{e-e^{-1}}{2}}{\frac{e+e^{-1}}{2}} = \frac{e-e^{-1}}{e+e^{-1}} = \frac{e+\frac{1}{e}}{e-\frac{1}{e}} = \frac{\frac{e^2+1}{e}}{\frac{e^2-1}{e}} = \frac{e^2+1}{e^2-1}$ =L.H.S

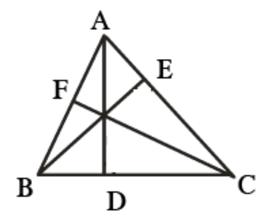
Part –III

Answer any SEVEN questions **7 x 2 = 14**

Question No. 40 is compulsory

31. Prove that the sum of the vectors directed from the vertices to the midpoints of opposite sides of a triangle is zero.

Let O be the point of reference. In ΔABC, let D, E, F be the midpoints of the sides BC, CA, AB respectively.



Then, $\overrightarrow{OD} = \frac{\overrightarrow{OB}+\overrightarrow{OC}}{2}$, $\overrightarrow{OE} = \frac{\overrightarrow{OC}+\overrightarrow{OA}}{2}$, $\overrightarrow{OF} = \frac{\overrightarrow{OA}+\overrightarrow{OB}}{2}$

Now $\overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CF} = \overrightarrow{OD} - \overrightarrow{OA} + \overrightarrow{OE} - \overrightarrow{OB} + \overrightarrow{OF} - \overrightarrow{OC}$

$= \frac{\overrightarrow{OB}+\overrightarrow{OC}}{2} - \overrightarrow{OA} + \frac{\overrightarrow{OC}+\overrightarrow{OA}}{2} - \overrightarrow{OB} + \frac{\overrightarrow{OA}+\overrightarrow{OB}}{2} - \overrightarrow{OC}$

$= \frac{2\overrightarrow{OA}+2\overrightarrow{OB}+2\overrightarrow{OC}}{2} - \overrightarrow{OA} - \overrightarrow{OB} - \overrightarrow{OC}$

$= \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} - \overrightarrow{OA} - \overrightarrow{OB} - \overrightarrow{OC}$

$= 0$

32. Find the co-efficient of x^5 in the expansion of $\left(x + \frac{1}{x^3}\right)^{17}$

In the expansion of $\left(x + \frac{1}{x^3}\right)^{17}$, the general term is

$T_{r+1} = 17C_r x^{17-r} \left(\frac{1}{x^3}\right)^r = 17C_r x^{17-r} x^{-3r} = 17C_r x^{17-4r}$

Let T_{r+1} be the term containing x^5

Then, $17 - 4r = 5$

$-4r = 5 - 17$

$-4r = -12$

$r = 3$

$T_{r+1} = T_{3+1}$

$= 17C_3 x^{17-4(3)}$

$= \frac{17\times16\times15}{1\times2\times3} x^{17-12}$

$= 680\, x^5$

$\therefore$ coefficient of $x^5 = 680$

33. Find 5 geometric means between 576 and 9

Let G_1, G_2, G_3, G_4, G_5 be 5 geometric means between $a = 576$ and $b = 9$

Let the common ratio be r

$G_1 = 576r$

$G_2 = 576r^2$

$G_3 = 576r^3$

$G_4 = 576r^4$

$G_5 = 576r^5$

$9 = 576r^6$

$r^6 = \frac{9}{576}$

$r = \left(\frac{9}{576}\right)^{\frac{1}{6}} = \left(\frac{1}{64}\right)^{\frac{1}{6}} = \left(\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\right)^{\frac{1}{6}}$

$r = \frac{1}{2}$

$G_1 = 576r = 576\left(\frac{1}{2}\right) = 288$

$G_2 = 576r^2 = 576\left(\frac{1}{4}\right) = 144$

$G_3 = 576r^3 = 576\left(\frac{1}{8}\right) = 72$

$G_4 = 576r^4 = 576\left(\frac{1}{16}\right) = 36$

$G_5 = 576r^5 = 576\left(\frac{1}{32}\right) = 18$

Hence 288, 144, 72, 36, 18 are the required G.M's between 576 and 9

34. The slope of one of the straight lines $ax^2 + 2hxy + by^2 = 0$ is twice that of the order, show that $8h^2 = 9ab$

Let the separate equation of $ax^2 + 2hxy + by^2 = 0$, $y = m_1x$ and $y = m_2x$

So that $m_1 + m_2 = -\frac{2h}{b}$

$m_1m_2 = \frac{a}{b}$

It is given that $m_1 = 2m_2$

$2m_2 + m_2 = -\frac{2h}{b}$

$3m_2 = -\frac{2h}{b}$

$m_2 = -\frac{2h}{3b}$

$m_1m_2 = \frac{a}{b}$

$2m_2.m_2 = \frac{a}{b}$

$2\left(-\frac{2h}{3b}\right)\left(-\frac{2h}{3b}\right) = \frac{a}{b}$

$\frac{8h^2}{9b^2} = \frac{a}{b}$

$8h^2 = \frac{9b^2a}{b}$

$8h^2 = 9ab$

35. Show that $\sin 20^0 \sin 40^0 \sin 80^0 = \frac{\sqrt{3}}{8}$

$$\sin A \sin B = \frac{1}{2}(\cos(A-B) - \cos(A+B))$$

$\sin 20^0 \sin 40^0 \sin 80^0 = \sin 20^0 \frac{1}{2}\{\cos 40^0 - \cos 120^0\}$

$= \frac{1}{2}\sin 20^0 \left\{\cos 40^0 + \frac{1}{2}\right\}$

$= \frac{1}{2}\sin 20^0 \cos 40^0 + \frac{1}{4}\sin 20^0$

$= \frac{1}{4}(\sin 60^0 - \sin 20^0) + \frac{1}{4}\sin 20^0$

$= \frac{1}{4}\sin 60^0 - \frac{1}{4}\sin 20^0 + \frac{1}{4}\sin 20^0$

$= \frac{1}{4}\sin 60^0$

$= \frac{1}{4}\left(\frac{\sqrt{3}}{2}\right)$

$= \frac{\sqrt{3}}{8}$

36. Let $f: R \to R$ be defined by $f(x) = 3x + 2$ find f^{-1} and show that $f \circ f^{-1} = f^{-1} \circ f = I$

$f(x) = 3x + 2$

$y = 3x + 2$

$x = \frac{y-2}{3}$

$f^{-1}(x) = \frac{x-2}{3}$

$f \circ f^{-1}(x) = f\left(\frac{x-2}{3}\right) = 3\left(\frac{x-2}{3}\right) + 2 = x - 2 + 2 = x$

$f^{-1} \circ f(x) = f^{-1}(3x+2) = \frac{3x+2-2}{3} = \frac{3x}{3} = x$

$f \circ f^{-1} = f^{-1} \circ f = I$

37. Evaluate: $\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{\sin^{-1} x}$

Let $\sin^{-1} x = y$

Then $x = \sin y$ & $y \to 0$ as $x \to 0$

$\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{\sin^{-1} x} = \lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{\sin^{-1} x} \times \left(\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$

$= \lim_{x\to 0} \frac{(1+x)-(1-x)}{\sin^{-1} x} \times \left(\frac{1}{\sqrt{1+x}+\sqrt{1-x}}\right)$

$$= \lim_{y\to 0}\left(\frac{2\sin y}{y}\right)\lim_{y\to 0}\left(\frac{1}{\sqrt{1+\sin^{-1}y}+\sqrt{1-\sin^{-1}y}}\right)$$

$$= 2\times 1\times \frac{1}{2} = 1$$

38. Integrate: $\int (3x+4)\sqrt{3x+7}dx$

$$\int (3x+4)\sqrt{3x+7}dx = \int\{3x+7-3\}\sqrt{3x+7}dx$$

$$= \int\{(3x+7)-3\}\sqrt{3x+7}dx$$

$$= \int\left\{(3x+7)\sqrt{3x+7} - 3\sqrt{3x+7}\right\}dx$$

$$= \int\left\{(3x+7)^{\frac{3}{2}} - 3(3x+7)^{\frac{1}{2}}\right\}dx$$

$$= \frac{1}{3}\frac{(3x+7)^{\frac{5}{2}}}{5/2} - 3\frac{1}{3}\frac{(3x+7)^{\frac{3}{2}}}{3/2} + c$$

$$= \frac{2}{15}(3x+7)^{\frac{5}{2}} - \frac{2}{3}(3x+7)^{\frac{3}{2}} + c$$

39. Two cards are drawn from a pack of 52 cards in succession. Find the Probability that both are kings when,
(i)The first drawn card is replaced (ii) The card is not replaced

Let A be the event of drawing a king in the first drawn

B be the event of drawing a king in the second drawn.

Case (i) Card is replaced

$n(A) = 4$ (King)

$n(B) = 4$ (King)

and $n(S) = 52$

Clearly the event A will not affect the probability of the occurrence of event B and therefore A and B are independent.

$$P(A\cap B) = P(A).P(B) = \frac{4}{52}\times\frac{4}{52} = \frac{1}{13}\times\frac{1}{13} = \frac{1}{169}$$

Case(ii) (Card is not replaced)

In the first drawn, there are 4 kings and 52 cards in total. Since the king drawn at the first drawn is not replaced, in the second drawn there are only 3 kings and 51 cards in total. Therefore the first event A affects the probability of the occurrence of the second event B.

$\therefore A$ and B are not independent they are dependent events.

$P(A \cap B) = P(A).P(B/A)$

$P(A) = \frac{4}{52}, P(B/A) = \frac{3}{51}$

$P(A \cap B) = P(A).P(B/A) = \frac{4}{52} \times \frac{3}{51} = \frac{1}{13} \times \frac{1}{17} = \frac{1}{221}$

40. Find k so that $A^2 = KA - 2I$ where $A = \begin{pmatrix} 3 & -2 \\ 4 & -2 \end{pmatrix}$

$A = \begin{pmatrix} 3 & -2 \\ 4 & -2 \end{pmatrix}$

$A^2 = \begin{pmatrix} 3 & -2 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} 3 & -2 \\ 4 & -2 \end{pmatrix} = \begin{pmatrix} 9-8 & -6+4 \\ 12-8 & -8+4 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 4 & -4 \end{pmatrix}$

Given $A^2 = KA - 2I$

$\begin{pmatrix} 1 & -2 \\ 4 & -4 \end{pmatrix} = k\begin{pmatrix} 3 & -2 \\ 4 & -2 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

$\begin{pmatrix} 1 & -2 \\ 4 & -4 \end{pmatrix} = \begin{pmatrix} 3k & -2k \\ 4k & -2k \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

$\begin{pmatrix} 1 & -2 \\ 4 & -4 \end{pmatrix} = \begin{pmatrix} 3k-2 & -2k \\ 4k & -2k-2 \end{pmatrix}$

$\therefore\ 1 = 3k - 2$

$3 = 3k$

$k = 1$

Part –IV

Answer all the questions **7 x 5 = 35**

41. Using factor method prove that

$$\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc(a+b+c)^3$$

Let $\Delta = \begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$ Put $a = 0$ in Δ

$\Delta = \begin{vmatrix} (b+c)^2 & 0 & 0 \\ b^2 & c^2 & b^2 \\ c^2 & c^2 & b^2 \end{vmatrix} = 0$ [$\because C_2$ is proportional to C_3]

$\therefore (a - 0)$ is a factor of Δ

Similarly on putting $b = 0, c = 0$ that the value of Δ is zero.

$\therefore a, b, c$ are factors of Δ.

Put $a+b+c=0$

$b+c=-a, \quad a+c=-b, \quad a+b=-c$

$$\Delta=\begin{vmatrix}(-a)^2 & a^2 & a^2\\ b^2 & (-b)^2 & b^2\\ c^2 & c^2 & (-c)^2\end{vmatrix}=0$$

Since three columns are identical, $(a+b+c)^2$ is a factor of Δ.

$\therefore abc(a+b+c)^2$ is a factors of Δ

The degree of factor is 5.

<u>To find m</u>:

The product of the leading diagonal elements $(b+c)^2(c+a)^2(a+b)^2$ is of degree 6.

$m=6-5=1$, If $m=1$

$\therefore$ The other factor of Δ must be $k(a+b+c)$

$$\begin{vmatrix}(b+c)^2 & a^2 & a^2\\ b^2 & (c+a)^2 & b^2\\ c^2 & c^2 & (a+b)^2\end{vmatrix}=k\,abc(a+b+c)^3$$

<u>To find k</u>:

Take the values $a=1, b=1$ and $c=1$

$$\begin{vmatrix}4 & 1 & 1\\ 1 & 4 & 1\\ 1 & 1 & 4\end{vmatrix}=k(1)(1)(3)^3$$

$54=27k$

$k=2$

$\therefore \Delta=2abc(a+b+c)^3$

(Or)

Examine whether the vectors $\vec{\imath}+3\vec{\jmath}+\vec{k}, 2\vec{\imath}-\vec{\jmath}-\vec{k}$ and $7\vec{\jmath}+5\vec{k}$ are coplanar.

Let $\vec{\imath}+3\vec{\jmath}+\vec{k}=x(2\vec{\imath}-\vec{\jmath}-\vec{k})+y(7\vec{\jmath}+5\vec{k})$

$\vec{\imath}+3\vec{\jmath}+\vec{k}=2x\vec{\imath}-x\vec{\jmath}-x\vec{k}+7y\vec{\jmath}+5y\vec{k}$

$\vec{\imath}+3\vec{\jmath}+\vec{k}=2x\vec{\imath}-x\vec{\jmath}+7y\vec{\jmath}-x\vec{k}+5y\vec{k}$

$\vec{i} + 3\vec{j} + \vec{k} = 2x\vec{i} + (-x + 7y)\vec{j} + (-x + 5y)\vec{k}$

Equating $\vec{i}$ th component

$$1 = 2x \Rightarrow x = \frac{1}{2}$$

Equating $\vec{j}$ th component

$$3 = -x + 7y$$

$$3 = -\frac{1}{2} + 7y$$

$$3 = \frac{-1+14y}{2}$$

$$6 = -1 + 14y$$

$$14y = 7$$

$$y = \frac{7}{14} = \frac{1}{2}$$

Equating $\vec{k}$ th component

$$1 = -x + 5y$$

Substituting x, y values in above equation

$$-x + 5y = -\frac{1}{2} + 5\left(\frac{1}{2}\right) = -\frac{1}{2} + \frac{5}{2} = \frac{4}{2} = 2 \neq 1$$

The third equation is not satisfied.

∴ They are not coplanar.

42. If $A + B = 45^0$, show that $(1 + \tan A)(1 + \tan B) = 2$ and hence deduce the value $\tan 22\frac{1}{2}^0$

Given $A + B = 45^0$

$$\tan(A + B) = \tan 45^0$$

$$\frac{\tan A + \tan B}{1 - \tan A . \tan B} = 1$$

$$\tan A + \tan B = 1 - \tan A . \tan B$$

$$1 + \tan A + \tan B = 2 - \tan A . \tan B \text{ (add 1 on both sides)}$$

$$1 + \tan A + \tan B + \tan A . \tan B = 2$$

$$1 + \tan A + \tan B\,(1 + \tan A) = 2$$

$$(1 + \tan A)\,(1 + \tan B) = 2$$

Take $A = B$, then $A + B = 45^0$

$$A + A = 45^0$$

$$2A = 45^0$$

$$A = 22\frac{1}{2}^0 = B$$

$$(1 + \tan A)^2 = 2$$

$$\therefore \left(1 + \tan 22\frac{1}{2}\right)^2 = 2$$

$$1 + \tan 22\frac{1}{2} = \pm\sqrt{2}$$

$$\tan 22\frac{1}{2} = \pm\sqrt{2} - 1$$

$22\frac{1}{2}$ is acute, $\tan 22\frac{1}{2}$ is positive.

$$\therefore \tan 22\frac{1}{2} = \sqrt{2} - 1$$

(Or)

State and Prove Napier's formula.

In any triangle ABC

(1) $\tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot\frac{C}{2}$

(2) $\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}$

(3) $\tan\frac{C-A}{2} = \frac{c-a}{c+a}\cot\frac{B}{2}$ are true

These are called Napier's formulae

(1) $\tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot\frac{C}{2}$

From sine formulae

$$\frac{a-b}{a+b}\cot\frac{C}{2} = \frac{2R\sin A - 2R\sin B}{2R\sin A + 2R\sin B}\cot\frac{C}{2}$$

$$= \frac{\sin A - \sin B}{\sin A + \sin B}\cot\frac{C}{2}$$

$$= \frac{2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)}{2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)}\cot\frac{C}{2}$$

$= \cot\left(\frac{A+B}{2}\right)\tan\left(\frac{A-B}{2}\right)\cot\frac{C}{2}$

$= \cot\left(90-\frac{C}{2}\right)\tan\left(\frac{A-B}{2}\right)\cot\frac{C}{2}$

$= \tan\frac{C}{2}\tan\left(\frac{A-B}{2}\right)\cot\frac{C}{2}$

$= \tan\left(\frac{A-B}{2}\right)$

$\therefore \tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot\frac{C}{2}$

Similarly, we can prove other two results (2),(3)

43. Prove by Mathematical Induction $1^2+2^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$, for all $n \in N$

Step 1:

Let $P(n)$ denote the statement $1^2+2^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$

Put $n=1$

$P(1)$ is the statement $1^2 = \frac{1(1+1)(2(1)+1)}{6}$

$1 = \frac{1(2)(3)}{6}$

$1 = \frac{6}{6}$

$1 = 1$

$\therefore P(1)$ is true.

Step 2:

Now assume that the statement be true for $n = k$

Assume $P(k)$ be true $1^2+2^2+\cdots+k^2 = \frac{k(k+1)(2k+1)}{6}$

Step 3:

To prove $1^2+2^2+\cdots+k^2+(k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$ is true.

$[1^2+2^2+\cdots+k^2]+(k+1)^2 = \frac{k(k+1)(2k+1)}{6}+(k+1)^2$

$= \frac{k(k+1)(2k+1)+6(k+1)^2}{6}$

$= \frac{(k+1)[k(2k+1)+6(k+1)]}{6}$

$= \frac{(k+1)[2k^2+k+6k+6]}{6}$

$= \frac{(k+1)[2k^2+7k+6]}{6}$

$$1^2 + 2^2 + \cdots + k^2 + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

$\therefore P(k+1)$ is true

Thus, if $P(k)$ is true, then $P(k+1)$ is true

Step 4:

By the principle of Mathematical induction, $P(n)$ is true for all $n \in N$

$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$, for all $n \in N$

(Or)

If a, b, c are in H.P. Prove that $\frac{b+a}{b-a} + \frac{b+c}{b-c} = 2$

a, b, c are in H.P. $b = \frac{2ac}{a+c}$ $\frac{b}{a} = \frac{2c}{a+c}$ $\frac{b+a}{b-a} = \frac{2c+a+c}{2c-a-c} = \frac{3c+a}{c-a}$ $b = \frac{2ac}{a+c}$ $\frac{b}{c} = \frac{2a}{a+c}$ $\frac{b+c}{b-c} = \frac{2a+a+c}{2a-a-c} = \frac{3a+c}{a-c}$ $\frac{b+a}{b-a} + \frac{b+c}{b-c} = \frac{3c+a}{c-a} + \frac{3a+c}{a-c}$ $= \frac{3c+a}{c-a} - \frac{3a+c}{c-a}$ $= \frac{3c+a-3a-c}{c-a}$ $= \frac{2c-2a}{c-a}$ $= \frac{2(c-a)}{c-a}$ $= 2$	**Aliter:** $b = \frac{2a}{a+c}$ $b(a+c) = 2ac$ L.H.S $= \frac{b+a}{b-a} + \frac{b+c}{b-c} = \frac{(b+a)(b-c)+(b+c)(b-a)}{(b-a)(b-c)}$ $= \frac{b^2-bc+ab-ac+b^2-ab+bc-ac}{b^2-bc-ab+ac}$ $= \frac{2b^2-2ac}{b^2-bc-ab-ac}$ $= \frac{2(b^2-ac)}{b^2-b(a+c)+ac}$ $= \frac{2(b^2-ac)}{b^2-2ac+ac}$ $[\because b(a+c) = 2ac]$ $= \frac{2(b^2-ac)}{(b^2-ac)}$ $= 2$

44. Find the equation of the circle passing through the points $(1, 1)$, $(2, -1)$ and $(3, 2)$

Let the equation of the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$

It passes through (1, 1)

$1^2 + 1^2 + 2g(1) + 2f(1) + c = 0$

$2 + 2g + 2f + c = 0$

$2g + 2f + c = -2$……………………………….(1)

It passes through $(2,-1)$

$2^2+(-1)^2+2g(2)+2f(-1)+c=0$

$4+1+4g-2f+c=0$

$5+4g-2f+c=0$

$4g-2f+c=-5$................................(2)

It passes through $(3,2)$

$3^2+2^2+2g(3)+2f(2)+c=0$

$9+4+6g+4f+c=0$

$6g+4f+c=-13$......................(3)

From (1) and (2)

(1) $\Rightarrow 2g+\cancel{2f}+c=-2$

(2) $\Rightarrow 4g-\cancel{2f}+c=-5$

$6g \quad +2c=-7$ ....(5)

From (2) and (3)

(2)×2 $\Rightarrow 8g-\cancel{4f}+2c=-10$

(3) $\Rightarrow 6g+\cancel{4f}+c=-13$

$14g \quad +3c=-23$................(6)

From (5) and (6)

(5) ×3 $\Rightarrow 18g+6c=-21$

(6) ×2 $\Rightarrow 28g+6c=-46$

$-10g \quad =25$

$g=\frac{25}{-10}=-\frac{5}{2}$

Sub $g=-\frac{5}{2}$ in (5)

$6\left(-\frac{5}{2}\right)+2c=-7$

$3(-5)+2c=-7$

$-15+2c=-7$

$2c=-7+15=8$

$2c=8$

$c = \frac{8}{2} = 4$

Sub values of g and c in (1)

$2\left(-\frac{5}{2}\right) + 2f + 4 = -2$

$-5 + 2f + 4 = -2$

$-1 + 2f = -2$

$2f = -1$

$f = -\frac{1}{2}$

Equation of the circle be $x^2 + y^2 + 2\left(-\frac{5}{2}\right)x + 2\left(-\frac{1}{2}\right)y + 4 = 0$

$$x^2 + y^2 - 5x - y + 4 = 0$$

(Or)

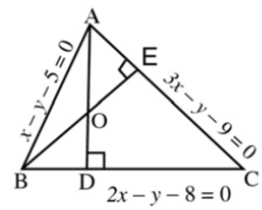
Find the co-ordinates of orthocenter of the triangle formed by the straight lines $x - y - 5 = 0$, $2x - y - 8 = 0$ and $3x - y - 9 = 0$

Let the equation of sides AB, BC and CA of a ΔABC be represented by

AB: $x - y - 5 = 0$ …………………..(1)

BC: $2x - y - 8 = 0$…………………..(2)

AC: $3x - y - 9 = 0$…………………(3)



Solving (1) and (3) $x = 2, y = -3$ $A(2, -3)$

The equation of the straight line BC is $2x - y - 8 = 0$.

The straight line perpendicular to it is of the form $x + 2y + k = 0$…………(4)

$A(2, -3)$ satisfies the equation (4) , $2 + 2(-3) + k = 0$

$$k = -4$$

The equation of AD is $x + 2y = -4$………………………..(5)

Solving the equations (1), (2) we get $B(3, -2)$

The straight line is perpendicular to $3x - y - 9 = 0$ is of the form

$$x + 3y + k = 0$$

But $B(3, -2)$ lies on this straight line $3 - 6 + k = 0$

$$-3 + k = 0$$

$$k = 3$$

The equation of BE is $x + 3y = -3$………………..(6)

Solving (5) and (6), we get the orthocenter O as $(-6, 1)$

45. If $y=\cos(\sin x)$, Prove that $\frac{d^2y}{dx^2}+\tan x\frac{dy}{dx}+y\cos^2 x=0$

$y=\cos\ (\sin x)$

$\frac{dy}{dx}=-\sin(\sin x)\times\frac{d}{dx}(\sin x)$

$=-\sin(\sin x)\times\cos x$

Again differentiating we get

$\frac{d^2y}{dx^2}=-[\sin(\sin x)\times(-\sin x)+\cos x\times(-\cos(\sin x)\times\cos x)]$

$=\sin(\sin x)\times(\sin x)-\cos x\times(\cos(\sin x)\times\cos x)$

$=\cos x\sin(\sin x)\times\frac{\sin x}{\cos x}-\cos x\times(\cos(\sin x)\times\cos x)$ (Multiply and divided by $\cos x$ in first term)

$=\left(-\frac{dy}{dx}\right)\times\tan x-\cos x\times(y\ \times\cos x)$

$=\left(-\frac{dy}{dx}\right)\times\tan x-y\cos^2 x$

$\frac{d^2y}{dx^2}=-\frac{dy}{dx}\times\tan x-y\cos^2 x$

$\frac{d^2y}{dx^2}+\tan x\frac{dy}{dx}+y\cos^2 x=0$

(OR)

For $\left|\frac{\Delta x}{a}\right|<1$ and for any rational index n, prove that $\lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}(a\neq 0)$

Put $\Delta x=x-a$ so that $\Delta x\to 0$ as $x\to a$ and $\left|\frac{\Delta x}{a}\right|<1$

$\therefore\frac{x^n-a^n}{x-a}=\frac{(a+\Delta x)^n-a^n}{\Delta x}=\frac{a^n\left(1+\frac{\Delta x}{a}\right)^n-a^n}{\Delta x}$

Applying Newton's Binomial Theorem for rational index we have

$\left(1+\frac{\Delta x}{a}\right)^n=1+nC_1\left(\frac{\Delta x}{a}\right)+nC_2\left(\frac{\Delta x}{a}\right)^2+\cdots nC_r\left(\frac{\Delta x}{a}\right)^r+\cdots$

$\frac{x^n-a^n}{x-a}=\frac{a^n\left[1+nC_1\left(\frac{\Delta x}{a}\right)+nC_2\left(\frac{\Delta x}{a}\right)^2+\cdots nC_r\left(\frac{\Delta x}{a}\right)^r+\cdots\right]-a^n}{\Delta x}$

$=\frac{a^n+nC_1\left(\frac{\Delta x}{a}\right)a^n+nC_2\left(\frac{\Delta x}{a}\right)^2a^n+\cdots nC_r\left(\frac{\Delta x}{a}\right)^r a^n+\cdots-a^n}{\Delta x}$

$=\frac{nC_1a^{n-1}\Delta x+nC_2a^{n-2}(\Delta x)^2+\cdots nC_ra^{n-r}(\Delta x)^r+\cdots}{\Delta x}$

$= nC_1 a^{n-1} + nC_2 a^{n-2}\,\Delta x + \cdots nC_r a^{n-r} (\Delta x)^{r-1} + \cdots$

$= nC_1 a^{n-1}$ + terms containing Δx and higher powers of Δx

Since $\Delta x = x - a, x \to a$ means $\Delta x \to 0$ and therefore

$\lim_{x\to a} \frac{x^n - a^n}{x-a} = \lim_{\Delta x \to 0} nC_1 a^{n-1} + \lim_{\Delta x\to 0}$ (terms containing Δx and higher powers of Δx)

$= nC_1 a^{n-1} + 0 + 0 + \cdots$

$= na^{n-1}$

46. Evaluate the definite integral as limit sum $\int_1^2 (2x+5)dx$

$\int_1^2 (2x+5)dx$

Let $f(x) = (2x+5)$ and $[a,b] = [1,2]$

$\Delta x = \frac{b-a}{n} = \frac{2-1}{n} = \frac{1}{n}$

$f(x) = 2x + 5$

$f(a + r\Delta x) = f\left(1 + r\frac{1}{n}\right) = 2\left(1 + \frac{r}{n}\right) + 5$

Let us divide the closed interval [1, 2] into n equal sub intervals of each length Δx

By the formula

$$\int_a^b f(x)dx = \lim_{\Delta x\to 0} \Delta x \sum_{r=1}^{n} f(a + r\Delta x)$$

$$\int_1^2 (2x+5)dx = \lim_{n\to\infty} \left(\frac{1}{n}\right)\sum_{r=1}^{n} 2\left(1+\frac{r}{n}\right) + 5$$

$$= \lim_{n\to\infty} \left(\frac{1}{n}\right)\sum_{r=1}^{n}\left(7 + \frac{2}{n}r\right)$$

$$= \lim_{n\to\infty} \left(\frac{1}{n}\right)\left[\sum_{r=1}^{n} 7 + \sum_{r=1}^{n}\frac{2}{n}r\right]$$

$$= \lim_{n\to\infty} \left(\frac{1}{n}\right)\left[7n + \frac{2}{n}.\frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to\infty} \left[7 + \frac{n+1}{n}\right]$$

$$= \lim_{n\to\infty} \left[7 + \left(1 + \frac{1}{n}\right)\right]$$

$$= 7 + 1$$

$\int_1^2 (2x+5)dx = 8$ square units

(OR)

Evaluate $\int \frac{3x+1}{\sqrt{2x^2+x+3}}\, dx$

$$3x+1 = A\frac{d}{dx}(2x^2+x+3)+B$$

$$3x+1 = A(4x+1)+B \text{.................}(1)$$

$$3x+1 = 4Ax+A+B$$

Equating like terms

$$4A = 3,\ \ A+B=1$$

$$A=\frac{3}{4},\ \ B=1-A=1-\frac{3}{4}=\frac{1}{4}$$

By (i) $3x+1=\frac{3}{4}(4x+1)+\frac{1}{4}$

$$\int \frac{3x+1}{\sqrt{2x^2+x+3}}\, dx = \int \frac{\frac{3}{4}(4x+1)+\frac{1}{4}}{\sqrt{2x^2+x+3}}\, dx$$

$$= \frac{3}{4}\int \frac{(4x+1)}{\sqrt{2x^2+x+3}}\, dx + \frac{1}{4}\int \frac{1}{\sqrt{2x^2+x+3}}\, dx$$

$$= \frac{3}{4}\left\{2\sqrt{2x^2+x+3}\right\}+I\text{....}(2) \qquad \left(\because \int \frac{f'(x)}{\sqrt{f(x)}}\, dx = 2\sqrt{f(x)}\right)$$

$$I = \frac{1}{4}\int \frac{1}{\sqrt{2x^2+x+3}}\, dx$$

$$= \frac{1}{4}\int \frac{\sqrt{4\times 2}}{\sqrt{(4x+1)^2+24.1}}\, dx$$

$$= \frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{(4x+1)^2+(\sqrt{23})^2}}\, dx$$

$$I = \frac{1}{\sqrt{2}}\left[\log(4x+1)+\sqrt{(4x+1)^2+(\sqrt{23})^2}\right]\times\frac{1}{4}$$

Substituting in (2) we get

$$\int \frac{3x+1}{\sqrt{2x^2+x+3}}\, dx = \frac{3}{2}\left\{\sqrt{2x^2+x+3}\right\}+\frac{1}{4\sqrt{2}}\left[\log(4x+1)+\sqrt{(4x+1)^2+(\sqrt{23})^2}\right]+c$$

47. If x is real, prove that the range of $f(x)=\frac{x^2-2x+4}{x^2+2x+4}$ is between $\left[\frac{1}{3}, 3\right]$

$y=\frac{x^2-2x+4}{x^2+2x+4}$

$yx^2+2xy+4y-x^2+2x-4=0$

$x^2(y-1)+x(2y+2)+(4y-4)=0$

This is a quadratic equation in x. It is given that x is real

Discriminent $\geq 0,\ B^2-4AC\geq 0$

$(2y+2)^2-4(y-1)(4y-4)\geq 0$

$4y^2 + 8y + 4 - 16y^2 + 16y + 16y - 16 \geq 0$

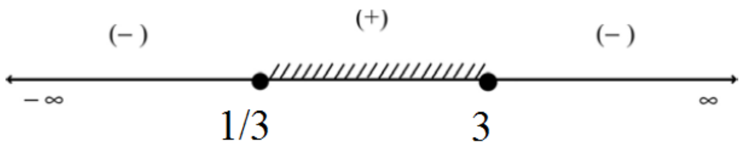
$-12y^2 + 40y - 12 \geq 0$

$3y^2 - 10y + 3 \leq 0$

$(3y - 1)(y - 3) \leq 0$

$\left(y - \frac{1}{3}\right)(y - 3) \leq 0$

$y \in \left[\frac{1}{3}, 3\right]$



$\therefore$ The range of $f(x) = \frac{x^2-2x+4}{x^2+2x+4}$ is between $\left[\frac{1}{3}, 3\right]$

(OR)

In a bolt factory A_1, A_2, A_3 manufacture respectively 25%, 35% and 40% of the total output of these 5, 4, 2 percent are defective bolts. A bolt is drawn at random from the product and is found to be defective. What is the probability that it was manufactured by machine A_2

Let E be the probability of drawing defective bolt.

Let, E_1, E_2 and E_3 be the event of drawing a bolt produced by the Machines A_1, A_2 and A_3 respectively. Then

$P(E_1) = \frac{25}{100}, P(E_2) = \frac{35}{100}, P(E_3) = \frac{40}{100},$

$P(E/E_1) = \frac{5}{100}, P(E/E_2) = \frac{4}{100}, P(E/E_3) = \frac{2}{100}$

$P(E_2/E) = \frac{P(E_2).P(E/E_2)}{P(E_1).P(E/E_1)+P(E_2).P(E/E_2)+P(E_3).P(E/E_3)}$

$= \frac{\frac{35}{100}\times\frac{4}{100}}{\left(\frac{25}{100}\times\frac{5}{100}\right)+\left(\frac{35}{100}\times\frac{4}{100}\right)+\left(\frac{40}{100}\times\frac{2}{100}\right)}$

$= \frac{35\times4}{(25\times5)+(35\times4)+(40\times2)}$

$= \frac{140}{125+140+80}$

$= \frac{140}{345}$

$P(E_2/E) = \frac{28}{69}$